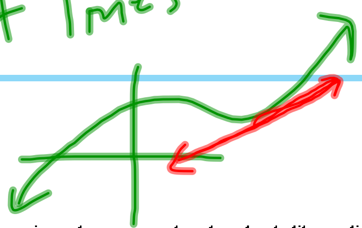
12.4 LINEAR FUNCTIONS

tangent lines



What is a Linear Function of Two Variables?

Linear functions played a central role in one-variable calculus because many one-variable functions have graphs that look like a line when we zoom in. In two-variable calculus, a *linear function* is one whose graph is a plane. In Chapter 14, we see that many two-variable functions have graphs which look like planes when we zoom in.

What Makes a Plane Flat?

What makes the graph of the function $z = f(x, y)$ a plane? Linear functions of *one* variable have straight line graphs because they ha constant slope. On a plane, the situation is a bit more complicated. If we walk around on a tilted plane, the slope is not always the san it depends on the direction in which we walk. However, at every point on the plane, the slope is the same as long as we choose the same direction. If we walk parallel to the x-axis, we always find ourselves walking up or down with the same slope; the same is true we walk parallel to the y-axis. In other words, the slope ratios $\Delta z/\Delta x$ (with y fixed) and $\Delta z/\Delta y$ (with x fixed) are each constant.

LINEAR FUNCTION, $f(x,y)$, is a plane.

$y = mx + b$

Example 1

A plane cuts the z-axis at $z = 5$, has slope 2 in the x direction and slope -1 in the y direction. What is the equation of the plane?

$z = 2x - 1y + 5$

$(0, 0, 5)$

Solution

Finding the equation of the plane means constructing a formula for the z-coordinate of the point on the plane directly above the point (x, y) in the xy-plane. To get to that point start from the point above the origin, where $z = 5$. Then walk x units in the x direction. Since the slope in the x direction is 2, the height increases by $2x$. Then walk y units in the y direction; since the slope in the y direction is -1, the height decreases by y units. Since the height has changed by $2x - y$ units, the z-coordinate is $5 + 2x - y$. Thus, the equation for the plane is

$$z = 5 + 2x - y.$$

$y = m(x - x_0) + y_0$

If a **plane** has slope m in the x direction, slope n in the y direction, and passes through the point (x_0, y_0, z_0), then its equation is

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

point-slope

This plane is the graph of the **linear function**

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0).$$

If we write $c = z_0 - mx_0 - ny_0$, then we can write $f(x, y)$ in the equivalent form

$$\boxed{f(x, y) = c + mx + ny.}$$

slope, intercept form

$y = mx + b$

A plane cuts the z-axis at z=-3. Moving in the y direction it has a slope of 3. Moving in the x direction it has a slope of -8. Find an equation of the plane.

$$\boxed{z = -3 + 3y - 8x}$$

Find an equation of the plane containing (1,2,3) with a slope of 2 in the x direction and a slope of -1 in the y direction.

$$z = z_0 + m(x - x_0) + n(y - y_0)$$

$$z = 3 + 2(x - 1) - 1(y - 2)$$

Find the equation of the plane passing through the points (1, 0, 1), (1, -1, 3), and (3, 0, -1).

Line

2pts

plane

3pts

$\Delta x = 0$

$\Delta y = -1$

$\Delta z = 2$

$\frac{\Delta z}{\Delta y} = -2$

$\cancel{\Delta x = 2}$

$\cancel{\Delta y = 1}$

$\Delta y = 0$

$\Delta x = 2$

$\Delta z = -2$

$\frac{\Delta z}{\Delta x} = -1$

$$\boxed{z = 1 - 1(x-1) - 2(y-0)}$$

Solution

The first two points have the same x-coordinate, so we use them to find the slope of the plane in the y-direction. As the y-coordinate changes from 0 to -1, the z-coordinate changes from 1 to 3, so the slope in the y-direction is $n = \Delta z/\Delta y = (3 - 1)/(-1 - 0) = -2$. The first and third points have the same y-coordinate, so we use them to find the slope in the x-direction; it is $m = \Delta z/\Delta x = (-1 - 1)/(3 - 1) = -1$. Because the plane passes through (1, 0, 1), its equation is

$$z = 1 - (x - 1) - 2(y - 0) \quad \text{or} \quad z = 2 - x - 2y.$$

You should check that this equation is also satisfied by the points (1, -1, 3) and (3, 0, -1).

$$R = 39700 + 79(d-200) + 239(f-100)$$

Table 12.10 *Revenue from Ticket Sales (Dollars)*

+ 100

		Full-price tickets (f)			
		100	200	300	400
Distance tickets (d)	200	39, 700	63, 600	87, 500	111, 400
	400	55, 500	79, 400	103, 300	127, 200
	600	71, 300	95, 200	119, 100	143, 000
	800	87, 100	111, 000	134, 900	158, 800
	1000	102, 900	126, 800	150, 700	174, 600

+200

Find R(f,d)

$$\frac{\Delta R}{\Delta d} = \frac{15800}{200} = 79 \qquad \frac{\Delta R}{\Delta f} = \frac{23900}{100} = 239$$

In every column, the revenue jumps by \$15, 800 for each extra 200 discount tickets. Thus, each column is a linear function of the number of discount tickets sold. In addition, every column has the same slope, 15,800/200 = 79 dollars/ticket. This is the price of a discount ticket. Similarly, each row is a linear function and all the rows have the same slope, 239, which is the price in dollars of a full-fare ticket. Thus, R is a linear function of f and d, given by:

$$R = 239f + 79d.$$

A **linear function** can be recognized from its table by the following features:

- Each row and each column is linear.
- All the rows have the same slope.
- All the columns have the same slope (although the slope of the rows and the slope of the columns are generally different).

The table contains values of a linear function. Fill in the blank and give a formula for the function.

$x\backslash y$	1.5	2.0
2	0.5	1.5
3	-0.5	?

$\Delta z = 1$

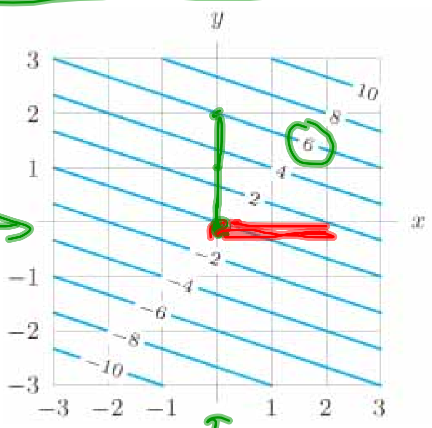
$-.5 + 1 = 0.5$

Solution

In the first column the function decreases by 1 (from 0.5 to -0.5) as x goes from 2 to 3. Since the function is linear, it must decrease by the same amount in the second column. So the missing entry must be 1.5 - 1 = 0.5. The slope of the function in the x-direction is -1. The slope in the y-direction is 2, since in each row the function increases by 1 when y increases by 0.5. From the table we get $f(2, 1.5) = 0.5$. Therefore, the formula is

$$f(x, y) = 0.5 - (x - 2) + 2(y - 1.5) = -0.5 - x + 2y.$$

Find the equation of the linear function whose contour diagram is in Figure 12.63.



$(0,0,0)$

$\frac{\Delta z}{\Delta y} = \frac{6}{2} = 3$

$\boxed{z = 0 + 1(x-0) + 3(y-0)}$

$\frac{\Delta z}{\Delta x} = \frac{2}{2} = 1$

Suppose we start at the origin on the $z = 0$ contour. Moving 2 units in the y direction takes us to the $z = 6$ contour; so the slope in the y direction is $\Delta z/\Delta y = 6/2 = 3$. Similarly, a move of 2 in the x-direction from the origin takes us to the $z = 2$ contour, so the slope in the x direction is $\Delta z/\Delta x = 2/2 = 1$. Since $f(0, 0) = 0$, we have $f(x, y) = x + 3y$.

MORE Examples

Which of the tables of values in Exercises 3, 4, 5 and 6 could represent linear functions?

3.

		y		
		0	1	2
	0	0	1	4
x	1	1	0	1
	2	4	1	0

4.

		y		
		0	1	2
	0	10	13	16
x	1	6	9	12
	2	2	5	8

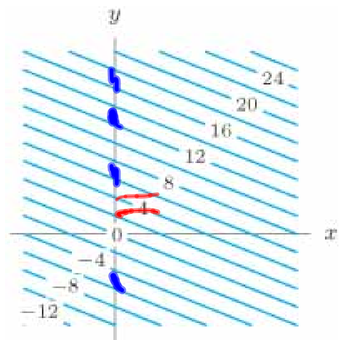
$$\frac{\Delta z}{\Delta x} = -4$$

$$\frac{\Delta z}{\Delta y} = \frac{3}{1}$$

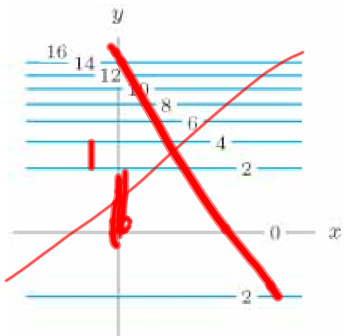
Which of the contour diagrams in Exercises 7 and 8 could represent linear functions?

7.

Contour lines are lines, evenly spaced



8.



$\frac{\Delta z}{\Delta y}$ NOT constant

9. Find the equation of the linear function $z = c + mx + ny$ whose graph contains the points (0, 0, 0), (0, 2, -1), and (-3, 0, -4).

$$\frac{\Delta z}{\Delta x} = \phi = \frac{-1}{0} = \phi$$

$$\frac{\Delta z}{\Delta y} = -\frac{1}{2}$$

$$\frac{\Delta z}{\Delta y} = \frac{-4}{0} = \phi$$

$$\frac{\Delta z}{\Delta x} = \frac{-4}{-3} = \frac{4}{3}$$

$$z = 0 + \frac{4}{3}(x-0) - \frac{1}{2}(y-0)$$

$$z = \frac{4}{3}x - \frac{1}{2}y$$

10. Find the linear function whose graph is the plane through the points (4, 0, 0), (0, 3, 0) and (0, 0, 2).

$z = c + mx + ny$

$$\frac{\Delta z}{\Delta x} = \frac{2}{-4} = -\frac{1}{2}$$

$$\frac{\Delta z}{\Delta y} = \frac{2}{-3}$$

$z = 2 - \frac{1}{2}x - \frac{2}{3}y$

12. Find the equation of the linear function $z = c + mx + ny$ whose graph intersects the xz-plane in the line $z = 3x + 4$ and intersects the yz-plane in the line $z = y + 4$.

$y = 0$

$x = 0$ $z = 4$

$(0, 0, 4)$

$y = 0$
$x = 0$
$z = 4$

$$\frac{\Delta z}{\Delta y} = 1, \quad \frac{\Delta z}{\Delta x} = 3$$

$$z = 4 + 3x + 1y$$

13. Suppose that z is a linear function of x and y with slope 2 in the x direction and slope 3 in the y direction.

(a) A change of 0.5 in x and -0.2 in y produces what change in z?

(b) If $z = 2$ when $x = 5$ and $y = 7$, what is the value of z when $x = 4.9$ and $y = 7.2$?

Δz Δz

(a) $(.5)(2) + (-.2)(3)$

$= 1 - .6$

$\boxed{\Delta z = .4}$

(b) $z = 2 + 2(x-5) + 3(y-7)$

$z = 2 + 2(-.1) + 3(.2)$

$= 2 - .2 + .6 \quad \boxed{= 2.4}$