$z = f(x,y)$

12.5 FUNCTIONS OF THREE VARIABLES

In applications of calculus, functions of any number of variables can arise. The density of matter in the universe is a function of three variables, since it takes three numbers to specify a point in space. Models of the US economy often use functions of ten or more variables. We need to be able to apply calculus to functions of arbitrarily many variables.

One difficulty with functions of more than two variables is that it is hard to visualize them. The graph of a function of one variable is a curve in 2-space, the graph of a function of two variables is a surface in 3-space, so the graph of a function of three variables would be a solid in 4-space. Since we can't easily visualize 4-space, we won't use the graphs of functions of three variables. On the other hand, it is possible to draw contour diagrams for functions of three variables, only now the contours are surfaces in 3-space.

graph
level
surfaces

$f(x,y,z) = C$

3 independent variables

4-d

✓✓✓ represent 4d - with concentric spheres

$$f(x,y,z) = x^2 + y^2 + z^2$$

$\downarrow$

$0 = x^2 + y^2 + z^2$ sphere $r=0$

$1 = x^2 + y^2 + z^2$ sphere $r=1$

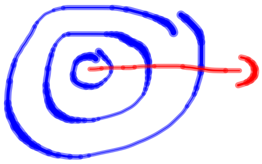
$4 = x^2 + y^2 + z^2$ sphere $r=2$

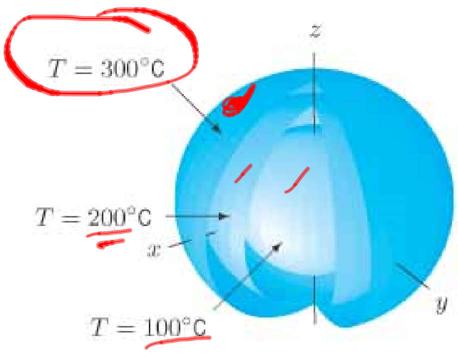
Representing a Function of Three Variables using a Family of Level Surfaces

A function of two variables, $f(x, y)$, can be represented by a family of level curves of the form $f(x, y) = c$ for various values of the constant, c.

A **level surface**, or **level set** of a function of three variables, $f(x, y, z)$, is a surface of the form $f(x, y, z) = c$, where c is a constant. The function f can be represented by the family of level surfaces obtained by allowing c to vary.

The temperature, in °C, at a point (x, y, z) is given by $T = f(x, y, z) = x^2 + y^2 + z^2$. What do the level surfaces of the function f look like and what do they mean in terms of temperature?

as  $T \uparrow$



z can be anything

What do the level surfaces of $f(x, y, z) = x^2 + y^2$ and $g(x, y, z) = z - y$ look like?

c

$1 = x^2 + y^2$

$25 = x^2 + y^2$

cylinder

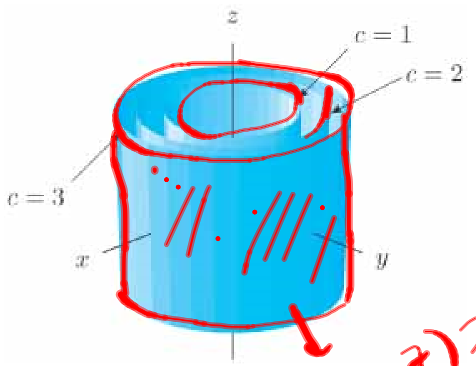
c

$z = y$

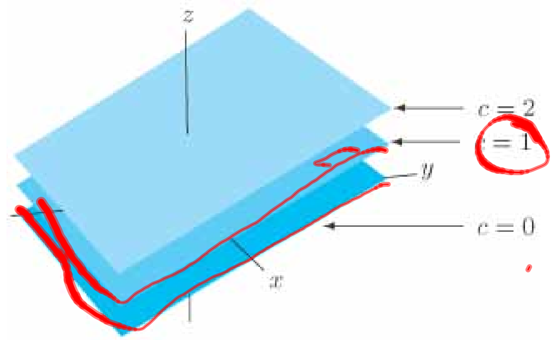
$0 = z - y$

$1 = z - y$

$z = 1 + y$



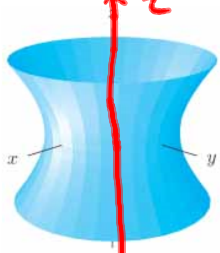
$f(x, y, z) = 3$



4-d 3-d $\boxed{\text{CONICS}}$

What do the level surfaces of $\boxed{f(x, y, z) = x^2 + y^2 - z^2}$ look like?

z



1 contour surface...

Figure 12.72 Hyperboloid of one sheet $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1}$

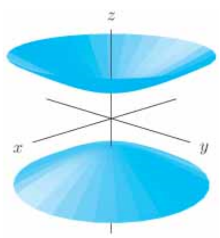


Figure 12.73 Hyperboloid of two sheets $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

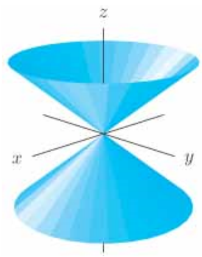


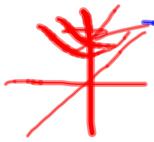
Figure 12.74 Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$

How Surfaces Can Represent Functions of Two Variables and Functions of Three Variables

You may have noticed that we have used surfaces to represent functions in two different ways. First, we used a *single* surface to represent a two-variable function $f(x, y)$. Second, we used a *family* of level surfaces to represent a three-variable function $g(x, y, z)$. These level surfaces have equation $g(x, y, z) = c$.

What is the relation between these two uses of surfaces? For example, consider the function

$$f(x, y) = x^2 + y^2 + 3.$$



Define

$$g(x, y, z) = x^2 + y^2 + 3 - z$$

The points on the graph of f satisfy $z = x^2 + y^2 + 3$, so they also satisfy $x^2 + y^2 + 3 - z = 0$. Thus the graph of f is the same as the level surface

$$g(x, y, z) = x^2 + y^2 + 3 - z = 0.$$

$c = 0 = x^2 + y^2 + 3 - z$

$z = x^2 + y^2 + 3$

c

level surface

$g(x, y, z) = 0$

A single surface that is the graph of a two-variable function $f(x, y)$ can be thought of as one member of the family of level surfaces representing the three-variable function

$$g(x, y, z) = f(x, y) - z.$$

The graph of f is the level surface $g = 0$.

$z = f(x, y)$ 3-d

$g(\cancel{x, y, z}) = f(x,y) - z$ 4-d

0

The graph of $f(x,y)$ is level surface when $g = 0$

3. Write the level surface $x + 2y + 3z = 5$ as the graph of a function $f(x, y)$.

constant [illegible]

$f(x, y, z) = x + 2y + 3z$

$\downarrow$

5

$x + 2y + 3z = 5$

$z = \dfrac{5 - x - 2y}{3} = f(x, y)$

4. Find a formula for a function $f(x, y, z)$ whose level surface $f = 4$ is a sphere of radius 2, centered at the origin.

$$x^2 + y^2 + z^2 = 2^2$$
$$= 4$$

$$f(x,y,z) = x^2 + y^2 + z^2$$

5. Write the level surface $x^2 + y + \sqrt{z} = 1$ as the graph of a function $f(x, y)$.

$$f(x, y, z) = x^2 + y + \sqrt{z} = 1$$

$$c = 1$$

$$\sqrt{z} = 1 - x^2 - y$$

$$z = f(x, y) = (1 - x^2 - y)^2$$

6. Find a formula for a function $f(x, y, z)$ whose level surfaces are spheres centered at the point (a, b, c).

$$f(x,y,z) = (x-a)^2 + (y-b)^2 + (z-c)^2$$