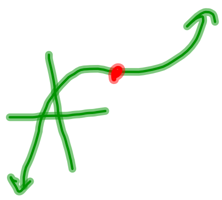


$f''(x)$ changes sign

$f''(x) = 0, \Phi$

POI

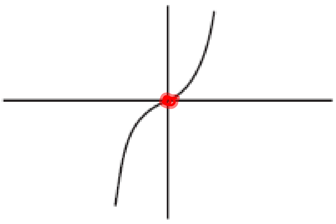
Points of Inflection

points of inflection.

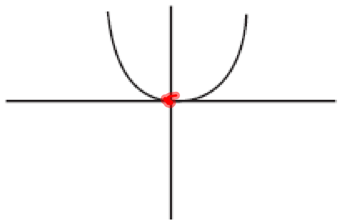
DOES NOT Δ sign

As an example of the four possible cases, we consider:

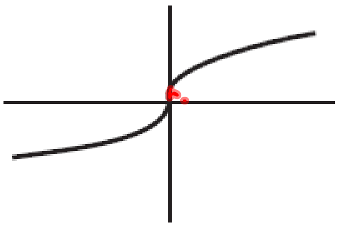
a) $f(x)=x^3$; $f''(x)=6x$, so $f''(0)=0$.



b) $f(x)=x^4$; $f''(x)=12x^2$, so $f''(0)=0$.

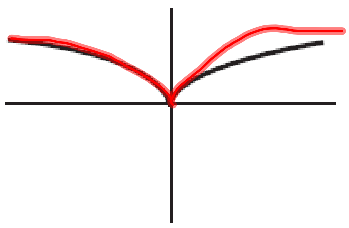


c) $f(x)=x^{1/3}$; $f''(x)=-\frac{2}{9}x^{-5/3}$, so $f''(0)$ does not exist.



$-\frac{2}{9}\cdot\frac{1}{x^{5/3}}$

d) $f(x)=x^{2/3}$; $f''(x)=-\frac{2}{9}x^{-4/3}$, so $f''(0)$ does not exist.



Cases a) and c) have inflection points at $x=0$, while cases b) and d) do not.

Example 2: *Finding Inflection Points*

Find the inflection points of $f(x) = x^4 + 2x^3 - 1$.

$f'(x) = 4x^3 + 6x^2$

$f''(x) = 12x^2 + 12x$

$= \underline{12x}(\underline{x-1})$

$x = 0, 1 \rightarrow f''(x) = 0$

x	$(-\infty, 0)$	0	$(0,1)$	1	$(1,\infty)$
f''	$+$	0	$-$	0	$+$

POI: $x = \{0, 1\}$

The Second Derivative Test

Example 3: *Applying the Second Derivative Test*

If the function f is defined by $f(x) = 2x^3 - 9x^2 + 27$, find the local maximum and minimum values by applying the Second Derivative Test.

$f'(x) = 6x^2 - 18x$

$= 6x(x-3)$

$c = 0, 3$

$f''(x) = 12x - 18$

$f''(0) = -18$

$f''(3) = 36 - 18 = 18$

$\boxed{\begin{array}{l} x=0,\ \text{max} \\ x=3,\ \text{min} \end{array}}$

The Second Derivative Test

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

1993 – AB 4, BC 3

4. Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \le x \le 2\pi$.

(a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

(b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

$f(\pi) = \ln(2 + \sin\pi) = \ln 2$ ✓

$f(2\pi) = \ln(2)$ ✓

$f'(x) = \dfrac{1}{2+\sin x} \cdot (\cos x)$

$\cos x = 0$, $x = 3\pi/2$ ✓

$2 + \sin x = 0$, $x \neq \emptyset$

$\sin x \neq -2$

$f(3\pi/2) = \ln\left(2 + \sin\left(\frac{3\pi}{2}\right)\right)$

$= \ln(2-1)$

$= \ln(1)$ ✓

Abs max $= \ln(2)$

Abs min $= \ln(1)$

$$f'(x) = \frac{1}{2+\sin x} \cdot (\cos x)$$

1993 – AB 4, BC 3

4. Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \le x \le 2\pi$.

(a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

(b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

$$f''(x) = \frac{(2+\sin(x))(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - (\sin^2 x + \cos^2 x)}{(2+\sin x)^2}$$

$\pi < x < 2\pi$ ✓

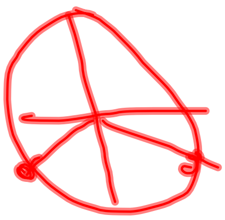
$$= \frac{-2\sin x - (\sin^2 x + \cos^2 x)}{(2 + \sin x)^2}$$

$$= \frac{-2\sin x - 1}{\cancel{(2+\sin x)^2}} = 0 = f''(0)$$

$$-2\sin x - 1 = 0$$

$$-2\sin x = 1$$

$$\sin x = -\tfrac{1}{2}$$ ✓



$$\boxed{\frac{7\pi}{6}, \quad \frac{11\pi}{6}}$$

1993 – AB 4, BC 3

4. Let f be the function defined by $f(x) = \ln(2 + \sin x)$ for $\pi \le x \le 2\pi$.

   (a) Find the absolute maximum value and the absolute minimum value of f. Show the analysis that leads to your conclusion.

   (b) Find the x-coordinate of each inflection point on the graph of f. Justify your answer.

---

a.) $f(x) = \ln(2 + \sin x) \ge 0$ since $2 + \sin x \ge 1$; $\pi \le x \le 2\pi$

$f'(x) = \dfrac{\cos x}{2 + \sin x}$; $f'(x) = 0$ $\quad \cos x = 0 \Rightarrow x = \dfrac{3\pi}{2}$

$f'(x)$: Q III $-$ | Q IV $+$ on x: π, $\dfrac{3\pi}{2}$, 2π

f-graph

x	$f(x)$
π	$\ln 2 = .693^{+}$
$\frac{3\pi}{2}$	$\ln 1 = 0$
2π	$\ln 2 = .693^{+}$

$\boxed{\text{f has an abs. min. of } \ln 1 = 0 \quad \text{f has an abs. max. of } \ln 2 \approx .693}$

b.) Inflection points occur at points where the direction of concavity changes.

$$f''(x) = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2 + \sin x)^2} = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

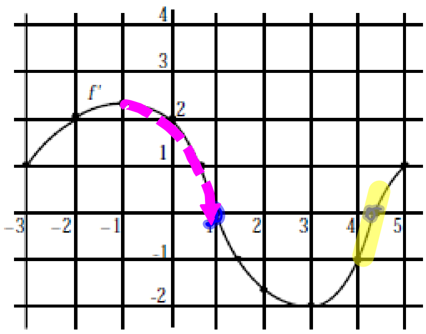
$f''(x) = 0$ when $\sin x = -\frac{1}{2} \Rightarrow x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

$f''(x)$: $-$ $+$ $-$ on x: π, $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, 2π

$\boxed{\text{f has inflection points at } x = \frac{7\pi}{6}, \frac{11\pi}{6}}$

Example Problems with POINTS OF INFLECTION

The graph of the derivative f' of a function f on the closed interval $[-3, 5]$ is shown below. With this graph as an aid, answer each of the following questions.



a) What are the critical numbers of f ? $x = 1, 4.3$

b) Determine the x-coordinate(s) of any local minimum point(s) of f.

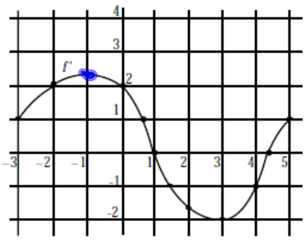
$X = 4.3$

c) On what interval is f both increasing and concave down?

$(-1, 0)$

d) Let another function g be defined by $g(x) = x^2 - 3x - 1$. With this information and the graph of f above, determine the value of the derivative of the composition $y = f(g(x))$ at $x = 3$.

The graph of the derivative f' of a function f on the closed interval $[-3, 5]$ is shown below. With this graph as an aid, answer each of the following questions.



$g'(3) = 2\cdot 3 - 3 = 3$

$g'(x) = 2x - 3$

$g(3) = 3^2 - 3(3) - 1 = -1$

d) Let another function g be defined by $g(x) = x^2 - 3x - 1$. With this information and the graph of f above, determine the value of the derivative of the composition $y = f(g(x))$ at $x = 3$.

$y' = f'(g(x)) \cdot g'(x)$

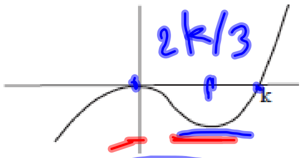
$y'(3) = f'(g(3)) \cdot g'(3)$

$= f'(-1)(3)$

$= (2.2)(3)$

$= 6.6$

The graph of $y = x^2(x-k)$ for a positive constant k looks something like the picture below.



a) Find the coordinates of the local minimum point in terms of k.

$y = x^3 - x^2 k$

$y' = 3x^2 - 2kx$

$= x(3x - 2k)$

$x = 0, \frac{2k}{3}$

$y = \left(\frac{2k}{3}\right)^3 - \left(\frac{2k}{3}\right)^2 \cdot k$

$= \frac{8k^3}{27} - \frac{4k^3}{9} \cdot \frac{3}{3}$

$= -\frac{4k^3}{27}$

3 Given the function: $f(x) = x \ln x$

Use your calculator for evaluating your function but do not use the graph of f(x) to justify your answers.

a) Find any critical points of f(x)

$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$

$= \ln x + 1 = 0$

$\ln(x) = -1$

e e^{-1}

b) Use the 1st Derivative Test to determine if the critical point is a relative min or max.

x	$x < 1/e$	$\frac{1}{e}$	$x > \frac{1}{e}$
f'	$-$	0	$+$

$x = \frac{1}{e}$

c) Now use the **2nd Derivative Test** to verify your result in a)

$y' = \ln x + 1$

$y'' = \frac{1}{x}$

$y'' = \frac{1}{1/e} = e$ $+$

$f' = 0$

$f'' > 0$

min

d) In the domain [0,4], what is the global minimum. Show your analysis.

$f\left(\frac{1}{e}\right) = \frac{1}{e} \ln\left(\frac{1}{e}\right)$

$x \ln x$

d) In the domain [0,4], what is the global minimum. Show your analysis.

$f\left(\frac{1}{e}\right) = \frac{1}{e} \ln\left(\frac{1}{e}\right)$

$= \frac{1}{e}(\ln(1) - \ln(e))$

$\boxed{= -\frac{1}{e}}$ global min

$f(0) = \text{undefined}$ $\quad 0 \ln(0)$

$f(4) = 4 \ln 4$

1 Given the function: $g(x) = \dfrac{x}{x^2 + 1}$ $\qquad \dfrac{1}{1+1} = \dfrac{1}{2}$

Use your calculator for evaluating your function but do not use the graph of f(x) to justify your answers.

a) Find any critical points of f(x) in the domain x > 0 $\qquad \boxed{x=1}$ ✓

$$f' = \frac{(x^2+1) - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

b) Use the 1st Derivative Test to determine if the critical point(s) is/are a relative min or max $\qquad \left(1, \frac{1}{2}\right)$ ✓

x	$x<1$	1	$x>1$
f'	$+$	0	$-$

max

d) In the domain [0,4], what is the global minimum. Show your analysis.

$$f(0) = \frac{0}{1} = 0 \quad ✓ \qquad \boxed{(0,0)}$$

$$f(4) = \frac{4}{16+1} = \frac{4}{17} \quad ✓$$