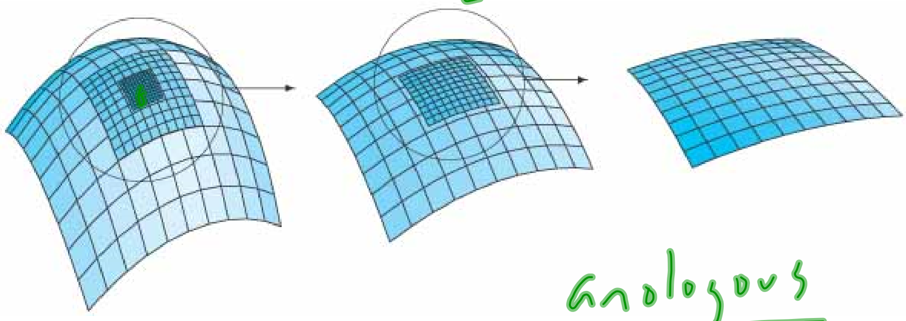
14.3 LOCAL LINEARITY AND THE DIFFERENTIAL

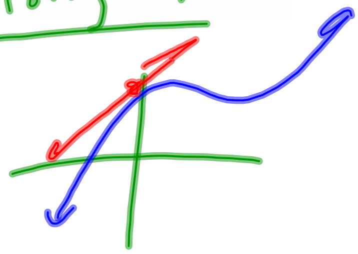
3d

zoom in

becomes a plane



analogous



Feb 18-6:57 PM

contour map

zoom in

zoom in more

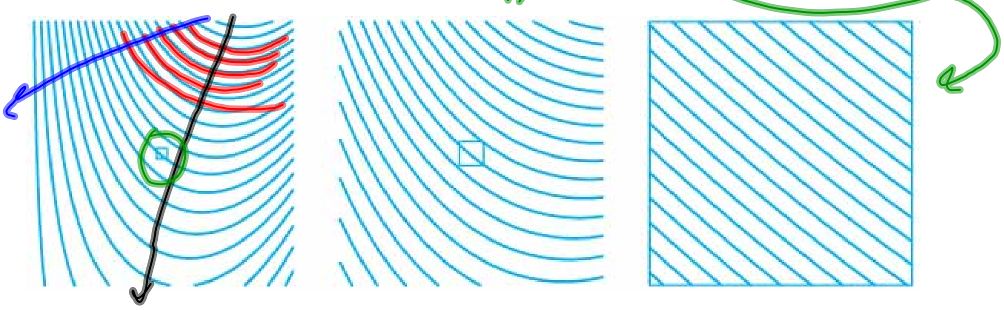


Figure 14.21 Zooming in on a contour diagram until the lines look parallel and equally spaced

Feb 18-6:57 PM

Zooming in on a table...

		$y=0$	$y=1$	$y=2$
x	1	1	2	9
	2	4	5	12
	3	9	10	17

		$y=0.9$	$y=1.0$	$y=1.1$
x	1.9	4.34	4.61	4.94
	2.0	4.73	5.00	5.33
	2.1	5.14	5.41	5.74

		$y=0.99$	$y=1.00$	$y=1.01$
x	1.99	4.93	4.96	4.99
	2.00	4.97	5.00	5.03
	2.01	5.01	5.04	5.07

Feb 18-6:58 PM

Zooming in Algebraically: Differentiability

Seeing a plane when we zoom in at a point tells us (provided the plane is not vertical) that $f(x, y)$ is closely approximated near that point by linear function, $L(x, y)$:

Linear

$$f(x, y) \approx L(x, y).$$

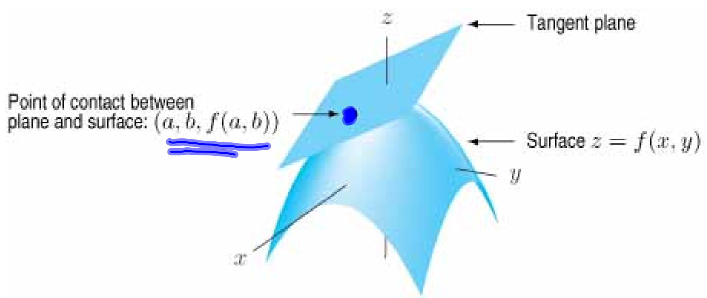


Figure 14.22 The tangent plane to the surface $z = f(x, y)$ at the point (a, b)

Feb 18-7:00 PM

Quick Review:

Find a tangent line to $y=x^3+x$ at $x=1$.

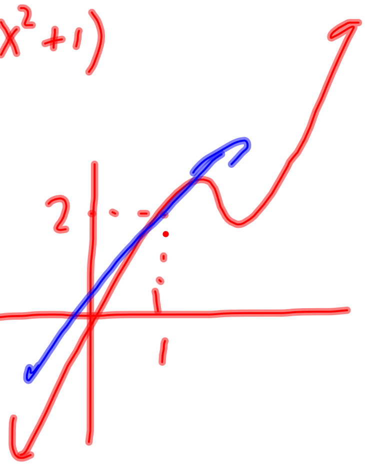
$x=1$

$y = 1^3+1 = 2$

$y = x(x^2+1)$

$y = m(x-x_1)+y_1$

$y = 4(x-1)+2$



$m = y'(1)$

$y' = 3x^2+1$

$y'(1) = 3+1 = 4$

Jan 24-7:30 PM

How do we find the equation of the tangent plane at a point (a,b)?

$f(x,y)$

① Find $f(a,b) = z_1$

② Find f_x

③ Find f_y

$L(x,y) = f_x(x-a)+f_y(y-b)+f(a,b)$

Tangent Plane to the Surface $z = f(x, y)$ at the Point (a, b)

Assuming f is differentiable at (a, b), the equation of the tangent plane is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Feb 18-7:01 PM

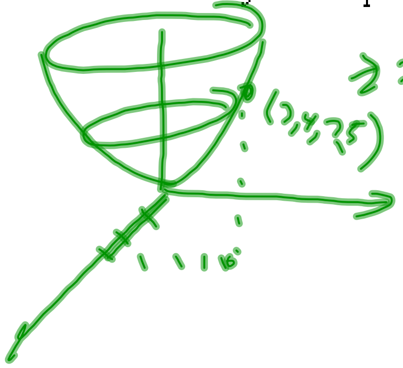
Find the equation for the tangent plane to the surface $z = x^2 + y^2$ at the point (3, 4).

$f_x = 2x$

$f_x(3,4) = 2 \cdot 3 = 6$

$f_y = 2y$

$f_y(3,4) = 2 \cdot 4 = 8$



$z = 3^2 + 4^2$
$= 25$

$$\boxed{L(x,y) = 6(x-3) + 8(y-4) + 25}$$

$$z = 25 + 6(x-3) + 8(y-4) = -25 + 6x + 8y.$$

Feb 18-7:02 PM

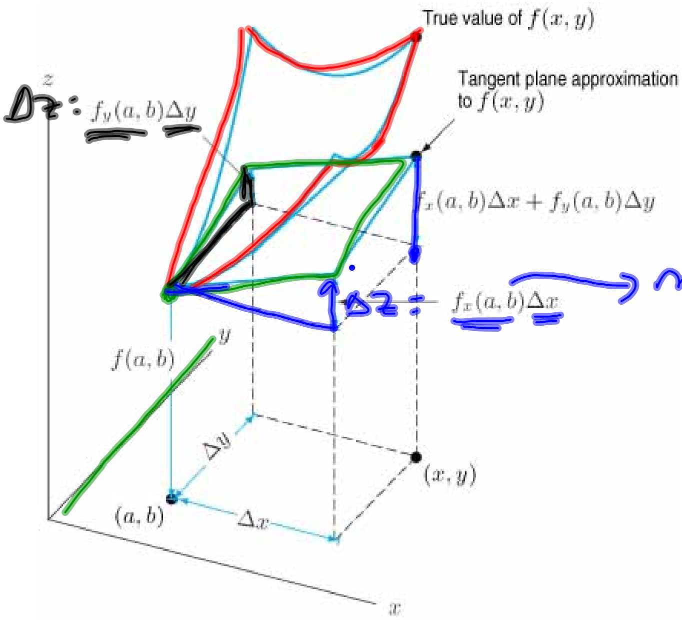
Tangent Plane Approximation to $f(x, y)$ for (x, y) Near the Point (a, b)

Provided f is differentiable at (a, b), we can approximate $f(x, y)$:

$$f(x,y) \approx \boxed{f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).}$$

We are thinking of a and b as fixed, so the expression on the right side is linear in x and y. The right side of this approximation is called the **local linearization** of f near $x = a, y = b$.

Feb 18-7:04 PM



$\Delta z = f_x \, \Delta x + f_y \, \Delta y$

$m \cdot \Delta x = \Delta y$

Feb 18-7:05 PM

Find the local linearization of $f(x, y) = x^2 + y^2$ at the point $(3, 4)$. Estimate $f(2.9, 4.2)$ and $f(2, 2)$ using the linearization and compare your answers to the true values.

$L(x,y) = 25 + 6(x-3) + 8(y-4)$

$L(2.9, 4.2) = 25 + 6(-.1) + 8(.2)$

$= 25 - .6 + 1.2$

$= 25.6$

$f(2.9, 4.2)$

$= 26.05$

$f(2,2)$

$= 2^2 + 2^2$

$= 8$

$L(2,2) = 25 + 6(-1) + 8(-2)$

$= 25 - 6 - 16$

$= 3$

Feb 18-7:05 PM

Designing safe boilers depends on knowing how steam behaves under changes in temperature and pressure. Steam tables, such as Table 14.5, are published giving values of the function $V = f(T, P)$ where V is the volume (in ft^3) of one pound of steam at a temperature T (in °F) and pressure P (in lb/in^2).

(a) Give a linear function approximating $V = f(T, P)$ for T near 500°F and P near 24 lb/in^2.

(b) Estimate the volume of a pound of steam at a temperature of 505°F and a pressure of 24.3 lb/in^2.

Table 14.5 *Volume (in Cubic Feet) of One Pound of Steam at Various Temperatures and Pressures*

	Pressure P (lb/in^2)			
Temperature T (°F)	20	22	24	26
480	27.85	25.31	23.19	21.39
500	28.46	25.86	23.69	21.86
520	29.06	26.41	24.20	22.33
540	29.66	26.95	24.70	22.79

$V_P = \frac{21.86 - 23.69}{2} = -.915$

$V_T = \frac{24.20 - 23.69}{20} = .0255$

$V(T,P) = 23.69 + .0255(T-500) - .915(P-24)$

$V(505, 24.3) = 23.69 + .0255(5) - .915(.3)$

$= 23.54$

Feb 18-7:06 PM

Solution

(a) We want the local linearization around the point $T = 500$, $P = 24$, which is

$$f(T, P) \approx f(500, 24) + f_T(500, 24)(T - 500) + f_P(500, 24)(P - 24).$$

We read the value $f(500, 24) = 23.69$ from the table.

Next we approximate $f_T(500, 24)$ by a difference quotient. From the $P = 24$ column, we compute the average rate of change between $T = 500$ and $T = 520$:

$$f_T(500, 24) \approx \frac{f(520, 24) - f(500, 24)}{520 - 500} = \frac{24.20 - 23.69}{20} = 0.0255.$$

Note that $f_T(500, 24)$ is positive, because steam expands when heated.

Next we approximate $f_P(500, 24)$ by looking at the $T = 500$ row and computing the average rate of change between $P = 24$ and $P = 26$:

$$f_P(500, 24) \approx \frac{f(500, 26) - f(500, 24)}{26 - 24} = \frac{21.86 - 23.69}{2} = -0.915.$$

Note that $f_P(500, 24)$ is negative, because increasing the pressure on steam decreases its volume. Using these approximations for the partial derivatives, we obtain the local linearization:

$$V = f(T, P) \approx 23.69 + 0.0255(T - 500) - 0.915(P - 24) \text{ ft}^3 \quad \text{for } T \text{ near } 500\,°F \text{ and } P \text{ near } 24 \text{ lb/in}^2.$$

(b) We are interested in the volume at $T = 505$°F and $P = 24.3$ lb/in^2. Since these values are close to $T = 500$°F and $P = 24$ lb/in^2, we use the linear relation obtained in part (a).

$$V \approx 23.69 + 0.0255(505 - 500) - 0.915(24.3 - 24) = 23.54 \text{ ft}^3.$$

Feb 18-7:07 PM

Local linear approximations for functions of three or more variables follow the same pattern as for functions of two variables. The local linearization of $f(x, y, z)$ at (a, b, c) is given by

$$f(x, y, z) \approx f(a, b, c) + f_x(a, b, c)(x-a) + f_y(a, b, c)(y-b) + f_z(a, b, c)(z-c).$$

Example: $f(x,y,z) = x^2 + xy + z$

@ $f(1,2,3) = 1^2 + 1\cdot 2 + 3$
$= 1+2+3 = 6$

$f_x = 2x+y = 2(1)+2 = 4$

$f_y = x = (1) = 1$

$f_z = 1 = 1$

$L(x,y,z) = 6 + 4(x-1) + 1(y-2) + 1(z-3)$

Feb 18-7:07 PM

original

$L(x,y)$

The Differential

$$f(x, y) \approx f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b)$$

$-f(a,b) \quad -f(a,b)$

$$\Delta f = f(x, y) - f(a, b)$$

Change in f

$$\Delta f \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y.$$

Feb 18-7:09 PM

$$\Delta f \approx f_x(a,b)\Delta x + f_y(a,b)\Delta y.$$

$\Delta x \to 0$ → dx $\Delta y \to 0$ → dy

$\Delta f \to 0$ → dz or df

The Differential of a Function $z = f(x, y)$

The **differential**, df (or dz), at a point (a, b) is the linear function of dx and dy given by the formula

$$df = f_x(a,b)\,dx + f_y(a,b)dy.$$

The differential at a general point is often written $df = f_x\,dx + f_y\,dy$.

$$dz = \underbrace{df}_{\text{change in } z, f} = \underbrace{f_x(a,b)dx}_{\Delta z \text{ from } x} + \underbrace{f_y(a,b)dy}_{\Delta z \text{ from } y}$$

Feb 18-7:10 PM

Compute the differentials of the following functions.

(a) $f(x, y) = x^2 e^{5y}$

(b) $z = x \sin(xy)$

(c) $f(x, y) = x\cos(2x)$

$f_x = 2xe^{5y}$

$f_y = x^2 \cdot e^{5y} \cdot 5$

(a)

$df = 2xe^{5y}dx + 5x^2e^{5y}dy$

$f_x = 1 \cdot \sin(xy) + x \cdot \cos(xy) \cdot y$

$= \sin(xy) + xy\cos(xy)$

$f_y = x \cdot \cos(xy) \cdot x$

$= x^2\cos(xy)$

$df = [\sin(xy) + xy\cos(xy)]dx + x^2\cos(xy)dy$

(a) Since $f_x(x, y) = 2xe^{5y}$ and $f_y(x, y) = 5x^2e^{5y}$, we have

$$df = 2xe^{5y}\,dx + 5x^2e^{5y}dy.$$

(b) Since $\partial z/\partial x = \sin(xy) + xy\cos(xy)$ and $\partial z/\partial y = x^2\cos(xy)$, we have

$$dz = (\sin(xy) + xy\cos(xy))dx + x^2\cos(xy)\,dy.$$

(c) Since $f_x(x, y) = \cos(2x) - 2x\sin(2x)$ and $f_y(x, y) = 0$, we have

$$df = (\cos(2x) - 2x\sin(2x))\,dx + 0\,dy = (\cos(2x) - 2x\sin(2x))\,dx.$$

Feb 18-7:12 PM

The differential for $\rho = f(T, P)$ is

$$d\rho = f_T(T, P)\ dT + f_P(T, P)\ dP = \frac{-0.5363P}{(T + 273.15)^2} dT + \frac{0.5363}{T + 273.15} dP.$$

Feb 18-7:13 PM

The density ρ (in g/cm^3) of carbon dioxide gas CO_2 depends upon its temperature T (in °C) and pressure P (in atmospheres). The ideal gas model for CO_2 gives what is called the state equation

$$\rho = \frac{0.5363P}{T + 273.15} = (.5363P)(T + 273.15)^{-1}$$

Compute the differential $d\rho$. Explain the signs of the coefficients of dT and dP.

$$f_P = (.5363)(T + 273.15)^{-1} = \frac{.5363}{T + 273.15}$$

$$f_T = (.5363P) \cdot (-1) \cdot (T + 273.15)^{-2} \cdot (1)$$

$$= \frac{-.5363P}{(T + 273.15)^2}$$

$$d\rho = \frac{.5363}{T + 273.15} dP - \frac{.5363P}{(T + 273.15)^2} dT$$

$dP > 0$, as $P\uparrow, \rho\uparrow$

$dT < 0$, as $T\uparrow, \rho\downarrow$

Jan 24-7:35 PM

Dec 28-12:30 PM