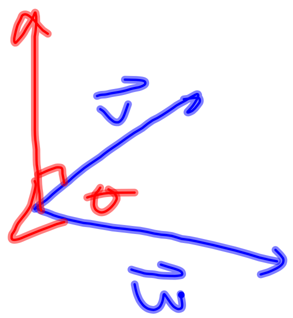
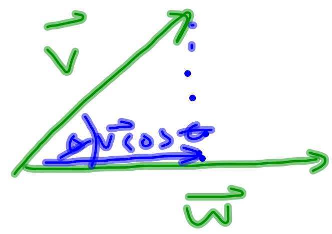
Cross-Product



$$\vec w \times \vec v = |\vec v|\,|\vec w|\sin\theta\;\vec n$$

Dot Product



$$\vec v \cdot \vec w = \|\vec v\cos\theta\| \cdot \|\vec w\|$$

Scalar

The Area of a Parallelogram

Consider the parallelogram formed by the vectors $\vec{v}$ and $\vec{w}$ with an angle of θ between them. Then Figure 13.35 shows

$$\text{Area of parallelogram} = \text{Base} \cdot \text{Height} = \|\vec{v}\| \, \|\vec{w}\| \sin\theta.$$

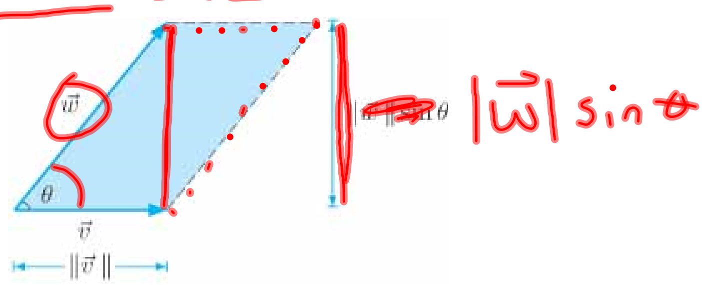


Figure 13.35 Parallelogram formed by $\vec{v}$ and $\vec{w}$ has Area $= \|\vec{v}\| \, \|\vec{w}\| \sin\theta$

$\|\vec{v} \times \vec{w}\|$ = Area of //gram defined by $\vec{v}, \vec{w}$

The right-hand rule: Place $\vec{v}$ and $\vec{w}$ so that their tails coincide and curl the fingers of your right hand through the smaller of the two angles from $\vec{v}$ to $\vec{w}$; your thumb points in the direction of the normal vector, $\vec{n}$.

$$\vec{v} \times \vec{w} = \boxed{|\vec{v}||\vec{w}|\sin\theta \cdot \vec{n}}$$

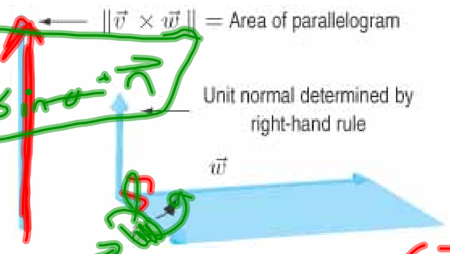


Figure 13.36 Area of parallelogram $= \|\vec{v} \times \vec{w}\|$

right hand on 1st vector, close fingers toward 2nd vector

magnitude, direction
↳ ⊥ to both vectors

The following two definitions of the **cross product** or **vector product** $\vec{v} \times \vec{w}$ are equivalent:

?

- ***Geometric Definition.*** If $\vec{v}$ and $\vec{w}$ are not parallel, then

$$\vec{v} \times \vec{w} = \begin{pmatrix} \text{Area of parallelogram} \\ \text{with edges } \vec{v} \text{ and } \vec{w} \end{pmatrix} \vec{n} = \boxed{(\|\vec{v}\| \, \|\vec{w}\| \sin\theta)\, \vec{n}},$$

  where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$ and $\vec{n}$ is the unit vector perpendicular to $\vec{v}$ and $\vec{w}$ pointing in the direction given by the right-hand rule. If $\vec{v}$ and $\vec{w}$ are parallel, then $\vec{v} \times \vec{w} = \vec{0}$.

- ***Algebraic Definition.***

$$\vec{v} \times \vec{w} = (v_2w_3 - v_3w_2)\,\vec{i} + (v_3w_1 - v_1w_3)\,\vec{j} + (v_1w_2 - v_2w_1)\,\vec{k}$$

  where $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

Helpful table Trick

$\vec{V} = V_1\vec{i} + V_2\vec{j} + V_3\vec{k}$

$\vec{W} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

$\vec{V} \times \vec{W}$

	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{V}$	V_1	V_2	V_3
$\vec{W}$	W_1	W_2	W_3

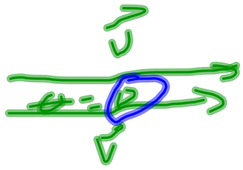
$(V_2W_3 - W_2V_3)\vec{i} - (V_1W_3 - W_1V_3)\vec{k}$

$+ (V_1W_2 - W_1V_2)\vec{j}$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\,\vec{i} + (v_3 w_1 - v_1 w_3)\,\vec{j} + (v_1 w_2 - v_2 w_1)\,\vec{k}\,.$$

$$= (v_1 w_3 - w_1 v_3)$$

For any vector $\vec{v}$, find $\vec{v} \times \vec{v}$.

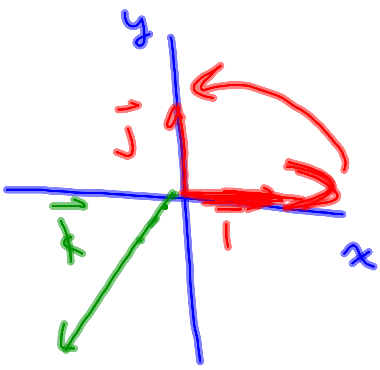


$$\vec{v} \times \vec{v} = \left[|\vec{v}|\,|\vec{v}| \sin\theta \right] \cdot \vec{n}$$

$\sin\theta \to 0$

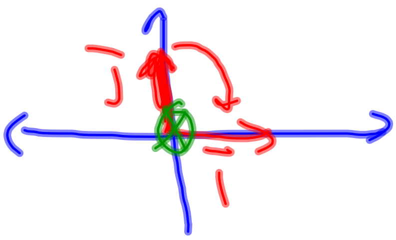
$$\boxed{\vec{v} \times \vec{v} = 0}$$

Find $\vec{i} \times \vec{j}$ and $\vec{j} \times \vec{i}$.



out of the page $\odot$

into the page $\otimes$



$\vec{j} \times \vec{i} = -\vec{k}$

$\vec{i} \times \vec{j} = \vec{k}$

Find the cross product of $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$ and $\vec{w} = 3\vec{i} + \vec{k}$ and check that the cross product is perpendicular to both $\vec{v}$ and $\vec{w}$.

$\vec{v} \times \vec{w}$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{array}$$

$$\vec{v} \times \vec{w} = (1\cdot 1 - (-2)(0))\vec{i} - (2\cdot 1 - 3(-2))\vec{j} + (2\cdot 0 - 3\cdot 1)\vec{k}$$

$$\vec{v} \times \vec{w} = \vec{i} - 8\vec{j} - 3\vec{k}$$

$$(\vec{v} \times \vec{w}) \cdot \vec{v} = 0$$

$$= (2\cdot 1) + (1(-8)) + (-2)(-3)$$

$$= 2 - 8 + 6$$

$$\to = 0 \checkmark$$

$$(\vec{v} \times \vec{w}) \cdot \vec{w} = (3\cdot 1) + (0\cdot(-8)) + (1(-3))$$

$$\to = 0 \checkmark$$

Properties of the Cross Product

For vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ and scalar λ

1. $\vec{w} \times \vec{v} = -(\vec{v} \times \vec{w})$ ✓
2. $(\lambda \vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}) = \vec{v} \times (\lambda \vec{w})$ λ is commutative
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

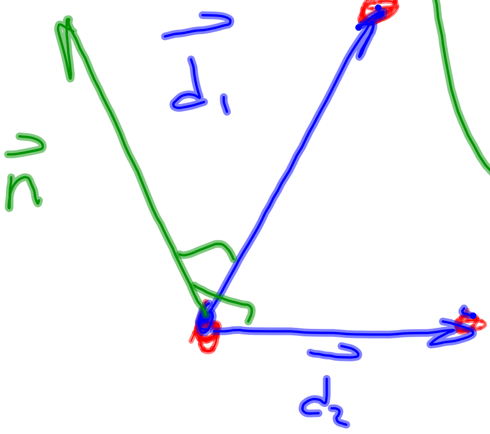
distribution property of cross product applies

The Equation of a Plane through Three Points

The equation of a plane is determined by a point $P_0 = (x_0, y_0, z_0)$ on the plane, and a normal vector, $\vec{n} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$:

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

Therefore, given three points, how could you use the cross product to find an equation of the plane?



$$\boxed{\vec{d_1} \times \vec{d_2}}$$

① get two vectors
② cross them
③ get $\vec{n}$
④ get Eqn for a plane

Find an equation of the plane containing the points $P = (1, 3, 0)$, $Q = (3, 4, -3)$, and $R = (3, 6, 2)$.

$\vec{PQ} = 2\vec{i} + \vec{j} - 3\vec{k}$

$\vec{n} = \vec{PQ} \times \vec{PR}$

$\vec{PR} = 2\vec{i} + 3\vec{j} + 2\vec{k}$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 2 & 3 & 2 \end{vmatrix} = (1\cdot 2 - 3(-3))\vec{i} - (2\cdot 2 - 2\cdot -3)\vec{j} + (2\cdot 3 + 2\cdot 1)\vec{k}$$

$\vec{n} = 11\vec{i} - 8\vec{j} + 8\vec{k}$ ✓✓

Eqn for Plane:

$$11(x-1) - 8(y-3) + 8(z-0) = 0$$

P1 (0,0,0)

Get help formatting numbers and formulas

Find the equation of a plane through the origin and perpendicular to $x - y + z = 2$ and $2x - 9y + 8z = 1$.

$P1 \perp (x - y + z = 2)$

hint: find the normal.

and you have a point.

$P1 \parallel \vec{i} - \vec{j} + \vec{k}$

If Plane A is perpendicular to plane B, then a normal line to plane B will be parallel to Plane A.

$P1 \parallel 2\vec{i} - 9\vec{j} + 8\vec{k}$

$P1 \perp \vec{n}$ $\quad \vec{n} = (\vec{i} - \vec{j} + \vec{k}) \times (2\vec{i} - 9\vec{j} + 8\vec{k})$

$$\vec{n} : \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 2 & -9 & 8 \end{vmatrix}$$

x-6*y-7*z= 0

$$(-8 + +9)\vec{i} - (8 - 2)\vec{j} + (-9 - -2)\vec{k}$$

$$\vec{n} \quad \vec{i} - 6\vec{j} - 7\vec{k}$$

$$EQN = 1(x-0) - 6(y-0) - 7(z-0) = 0$$

$$x - 6y - 7z = 0$$

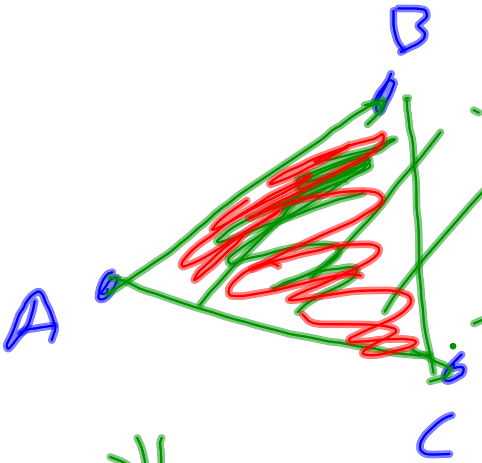
Area of a Triangle

Chapter 13, Section 13.4, Question 44

Find the vector representing the area of the triangle ABC where $A = (6, 7, 8)$, $B = (8, 6, 7)$, and $C = (7, 6, 8)$, oriented so that it faces upward.

$\overrightarrow{AB} = 2\vec{i} - \vec{j} - \vec{k}$

$\overrightarrow{AC} = \vec{i} - \vec{j} + 0\vec{k}$



$\overrightarrow{AB} \times \overrightarrow{AC}$

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & -1 \\ 1 & -1 & 0 \end{array}$$

$$\frac{\|\overrightarrow{AD} \times \overrightarrow{AC}\|}{2}$$

$$\frac{(-1\cdot 0 - 1)\vec{i} - (2\cdot 0 - 1)\vec{j} + (2(-1) - (1)(-1))\vec{k}}{2}$$

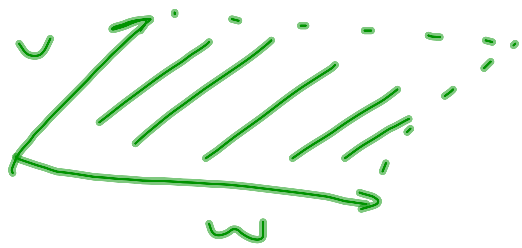
$\vec{v} = -\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} - \frac{1}{2}\vec{k}$

$\vec{A} = -\vec{v} = \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} + \frac{1}{2}\vec{k}$

triangle = $\frac{1}{2}$ parallelogram

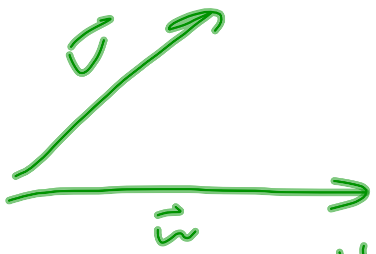
Area of a Parallelogram with edges $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ is given by

$$\text{Area} = \underline{\underline{\|\vec{v}\times\vec{w}\|}}, \qquad \text{where} \quad \vec{v}\times\vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$



$\|\vec{v}\times\vec{w}\| = \text{Area}$

Find the area of the parallelogram with edges $\vec{v} = 2\vec{i} + \vec{j} - 3\vec{k}$ and $\vec{w} = \vec{i} + 3\vec{j} + 2\vec{k}$.



$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix}$$

$$= (2-(-9))\vec{i} - (4+3)\vec{j} + (6-1)\vec{k}$$

$$\|\vec{A}\| = \|11\vec{i} - 7\vec{j} + 5\vec{k}\|$$

$$= \sqrt{11^2 + (-7)^2 + 5^2}$$

$$= \sqrt{121 + 49 + 25}$$

$$= \sqrt{195}$$

Volume of a Parallelepiped

$$(\vec{b} \times \vec{c}) \cdot \vec{a} \rightarrow \text{scalar} \downarrow \text{volume}$$

Area of base of parallelepiped $= \| \vec{b} \times \vec{c} \|$.

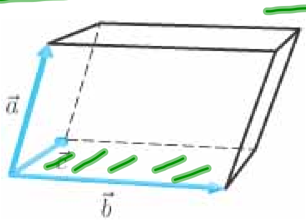
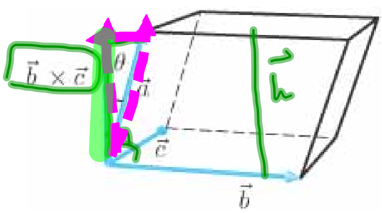


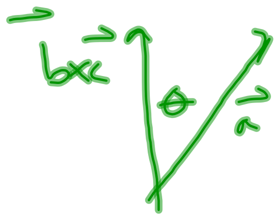
Figure 13.39 Volume of a Parallelepiped

$$= (\vec{b} \times \vec{c}) \cdot \vec{a} \cos\theta$$

$$= \| \vec{b} \times \vec{c} \| \, \| \vec{a} \| \cdot \cos\theta$$

what is the height? $a \cos \vec{\theta}$





what is the equation for the volume??

Volume of a parallelepiped with edges $\vec{a}$, $\vec{b}$, $\vec{c}$ is given by

$$\text{Volume} = \left|(\vec{b} \times \vec{c}) \cdot \vec{a}\right| = \text{Absolute value of the determinant} \left|\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right|$$

$$\left[(b_2c_3 - c_2b_3)\vec{i} - (b_1c_3 - c_1b_3)\vec{j} + (b_1c_2 - c_1b_2)\vec{k}\right]$$

$$\cdot(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$$

$$\left[a_1(b_2c_3 - c_2b_3) + (-b_1c_3 + c_1b_3)a_2 + (b_1c_2 - c_1b_2)a_3\right]$$

Volume

$\vec{b} \times \vec{c} \cdot \vec{a}$

From where?

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

Find an equation for the plane through the points in Exercises 12 and 13.

A B C

12. (1, 0, 0), (0, 1, 0), (0, 0, 1).

13. (3, 4, 2), (-2, 1, 0), (0, 2, 1).

$\vec{AB} = -\vec{i} + \vec{j} + 0\vec{k}$

$\vec{AC} = -\vec{i} + 0\vec{j} + \vec{k}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$\vec{AB} \times \vec{AC} = \vec{n} = (1)\vec{i} - (-1)\vec{j} + (0 - -1)\vec{k}$

$\vec{n} = \vec{i} + \vec{j} + \vec{k}$

EQN: $1(x-1) + 1(y-0) + 1(z-0) = 0$

$x + y + z = 0$