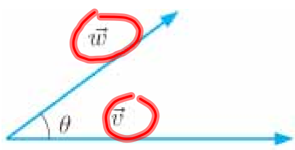
# 13.3 THE DOT PRODUCT




$\vec{v} \cdot \vec{w} = \#$

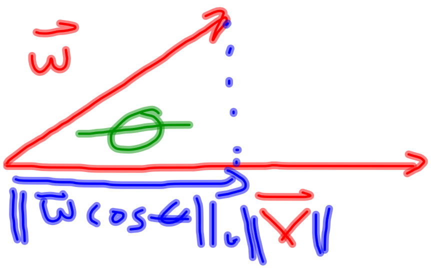
"scalar"

The following two definitions of the **dot product**, or **scalar product**, $\vec{v} \cdot \vec{w}$, are equivalent:

*

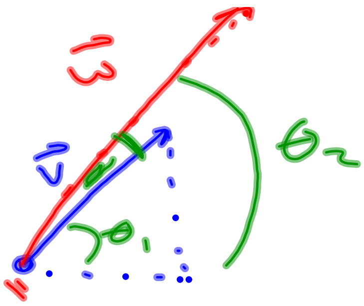
- ***Geometric Definition.*** $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$ where $\theta$ is the angle between $\vec{v}$ and $\vec{w}$ and $0 \le \theta \le \pi$.
- ***Algebraic Definition.*** $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$.

Notice that the dot product of two vectors is a *number*.




* multiply the mag of $\vec{w}$ in the $\vec{v}$ direction by the magnitude of $\vec{v}$.

Find the dot product using both methods of
v*w if
v=3i+4j and w=5i+12j




$\vec{v} \cdot \vec{w} = (3\cdot 5) + (4\cdot 12)$

Algebraic $= (15) + 48$

$= 63$

$\|\vec{v}\| \cos\theta \|\vec{w}\|$

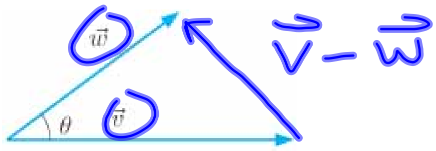
$\theta = \theta_2 - \theta_1$

$= \tan^{-1}\left(\frac{12}{5}\right) - \tan^{-1}\left(\frac{4}{3}\right) = 14.25^{\circ}$

Geometric $= \|\vec{v}\| \cdot \|\vec{w}\| \cos(14.25) = (5)(13)\cos 14.25 = 63$

**The law of cosines**

$$c^2 = a^2 + b^2 - 2ab\cos C$$




$$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta.$$

$$\|\vec{v}\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\|\vec{w}\|^2 = w_1^2 + w_2^2 + w_3^2$$

$$\begin{aligned}\|\vec{v} - \vec{w}\|^2 &= (v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2 \\ &= v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 + v_3^2 - 2v_3w_3 + w_3^2.\end{aligned}$$

$$v_1^2 - 2v_1w_1 + w_1^2 + v_2^2 - 2v_2w_2 + w_2^2 + v_3^2 - 2v_3w_3 + w_3^2. \quad = v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta.$$


$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta$$
$$\vec{w}\cdot\vec{v} = \|\vec{w}\|\,\|\vec{v}\|\cos\theta$$

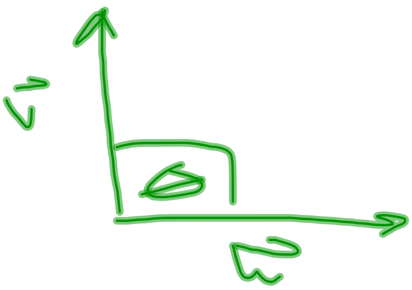
**Properties of the Dot Product**. For any vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ and any scalar $\lambda$,

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
2. $\vec{v} \cdot (\lambda \vec{w}) = \lambda(\vec{v} \cdot \vec{w}) = (\lambda \vec{v}) \cdot \vec{w}$
3. $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$

$$(2\vec{v})\cdot\vec{w} = 2(\vec{v}\cdot\vec{w}) = (2\cdot\vec{w})\cdot\vec{v}$$

Two nonzero vectors $\vec{v}$ and $\vec{w}$ are **perpendicular**, or **orthogonal**, if and only if

$$\vec{v} \cdot \vec{w} = 0.$$




$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cdot\cos\theta$$
$$= (\qquad)\cos 90$$
$$= 0$$

① $\vec{v} \cdot \vec{v}$ $\Rightarrow \theta = 0$ $= \| \vec{v} \|^2.$

$= \|\vec{v}\| \, \|\vec{v}\| \cos(0)$

$= |\vec{v}|^2$

② $\vec{i} \cdot \vec{k}$ $0$

③ $\vec{j} \cdot \vec{k}$ $z$ $y$ $0$ $= 0.$

④ $\vec{i} \cdot \vec{j}$ $y$ $x$ $0$

$\vec{a} \cdot \vec{b} = \|a\|\,\|b\| \cos\theta$

Suppose the vector $\vec{b}$ is fixed and has length 2; the vector $\vec{a}$ is free to rotate and has length 3. What are the maximum and minimum values of the dot product $\vec{a} \cdot \vec{b}$ as the vector $\vec{a}$ rotates through all possible positions? What positions of $\vec{a}$ and $\vec{b}$ lead to these values?

$\theta$

$|\vec{b}| = 2$

max

$\cos\theta = 1$

$\theta = 0$

$\vec{a} \cdot \vec{b} = 2 \cdot 3 \cdot \cos 0$

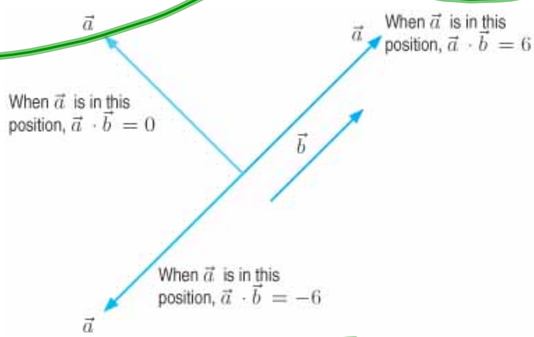
$= 6$

$\theta = \pi$

$\cos \pi = -1$

$\vec{a} \cdot \vec{b} = 2 \cdot 3 \cdot (-1)$

$= -6$




$|\vec{a} \cdot \vec{b}|$ is a min

@ $\theta = \pi/2,\ 3\pi/2$

$\cos(\pi/2) = \cos(\tfrac{3\pi}{2}) = 0$

Which pairs from the following list of 3-dimensional vectors are perpendicular to one another?

$$\vec{u} = \vec{i} + \sqrt{3}\,\vec{k}, \quad \vec{v} = \vec{i} + \sqrt{3}\,\vec{j}, \quad \vec{w} = \sqrt{3}\,\vec{i} + \vec{j} - \vec{k}.$$

$$\vec{u}\cdot\vec{v} = (1\cdot 1) + (0\cdot\sqrt{3}) + (\sqrt{3}\cdot 0)$$

$$\cancel{\vec{u}\cdot\vec{v}} = 1$$

$$\vec{u}\cdot\vec{w} : (1\cdot\sqrt{3}) + (0\cdot 1) + (\sqrt{3}\cdot(-1)$$

$$\vec{u}\cdot\vec{w} = 0$$

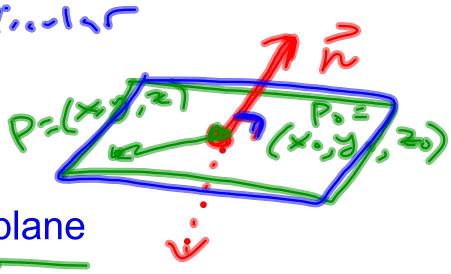
$$\vec{u}\perp\vec{w}$$

$$\cancel{\vec{v}\cdot\vec{w}} = (1\cdot\sqrt{3}) + (\sqrt{3}\cdot 1) + (0\cdot -1)$$

$$\neq 0$$

perpendicular

## Normal Vectors and the Equation of a Plane




$\vec n = a\vec i + b\vec j + c\vec k$ a vector normal to the plane

$P_0 = (x_0, y_0, z_0)$ a fixed point in the plane

$P = (x, y, z)$ a variable point in the plane

$\overrightarrow{P_0P} = (x - x_0)\,\vec i + (y - y_0)\,\vec j + (z - z_0)\,\vec k$ a vector contained in the plane

$\vec n \cdot \overrightarrow{P_0P} = 0.$

$\vec n \cdot \overrightarrow{P_0P} = \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$

normal line,

point on the plane

The **equation of the plane** with normal vector $\vec{n} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$ and containing the point $P_0 = (x_0, y_0, z_0)$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

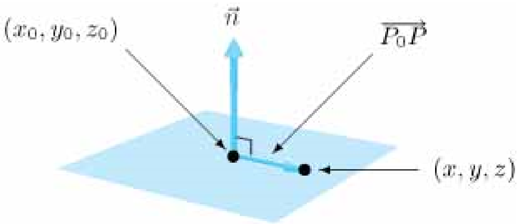
Letting $d = ax_0 + by_0 + cz_0$ (a constant), we can write the equation of the plane in the form

$$ax + by + cz = d.$$

$$ax - ax_0 + by - by_0 + cz - cz_0 = 0$$

$$ax + by + cz - d = 0$$

$$\boxed{ax + by + cz = d}$$



Find the equation of the plane perpendicular to $-\vec{i} + 3\vec{j} + 2\vec{k}$ and passing through the point (1, 0, 4).

$n = -\vec{i} + 3\vec{j} + 2k$ $P_0 : (1, 0, 4)$

Plane $-(x-1) + 3(y-0) + 2(z-4) = 0$

$-x + 1 + 3y + 2z - 8 = 0$

$\boxed{-x + 3y + 2z = 7}$

$$-(x-1) + 3(y-0) + 2(z-4) = 0,$$

$$-x + 3y + 2z = 7.$$

$n = a\vec{i} + b\vec{j} + c\vec{k}$  P: $a(x-x_0) + b(y-y_0) + c(z-z_0)$

Find a normal vector to the plane with equation

(a) $x - y + 2z = 5$

(b) $z = 0.5x + 1.2y$.

$1x - 1y + 2z = 5$

$\vec{n}: \vec{i} - \vec{j} + 2\vec{k}$

b) $0 = .5x + 1.2y - z$

$\vec{n} = 0.5\vec{i} + 1.2\vec{j} - \vec{k}$

(a) Since the coefficients of $\vec{i}$, $\vec{j}$, and $\vec{k}$ in a normal vector are the coefficients of $x$, $y$, $z$ in the equation of the plane, a normal vector is $\vec{n} = \vec{i} - \vec{j} + 2\vec{k}$.

(b) Before we can find a normal vector, we rewrite the equation of the plane in the form

$$0.5x + 1.2y - z = 0.$$

Thus, a normal vector is $\vec{n} = 0.5\vec{i} + 1.2\vec{j} - \vec{k}$.

## The Dot Product in $n$ Dimensions

The algebraic definition of the dot product can be extended to vectors in higher dimensions.

If $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ then the dot product of $\vec{u}$ and $\vec{v}$ is the **scalar**

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \ldots + u_n v_n.$$

4-d

A video store sells videos, tapes, CDs, and computer games. We define the quantity vector $\vec{q} = (q_1, q_2, q_3, q_4)$, where $q_1, q_2, q_3, q_4$ denote ~~the quantities sold of each~~ of the items, and the price vector $\vec{p} = (p_1, p_2, p_3, p_4)$, where $p_1, p_2, p_3, p_4$ denote the price per unit of each item. What does the dot product $\vec{p} \cdot \vec{q}$ represent?

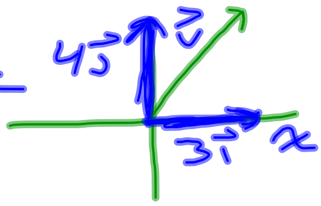
$$\vec{p} \cdot \vec{q} = q_1 p_1 + q_2 p_2 + q_3 p_3 + q_4 p_4$$

Rev: videos Rev: tapes

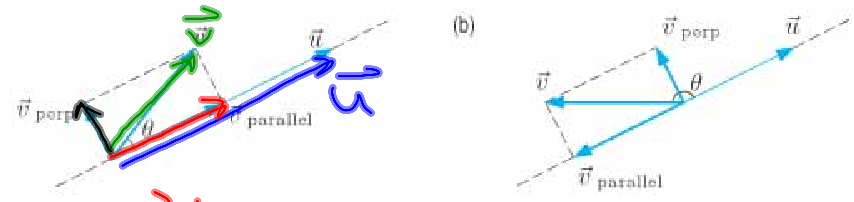
total Review:

## Resolving a Vector into Components: Projections

$\vec{v} = 3\vec{i} + 4\vec{j}$

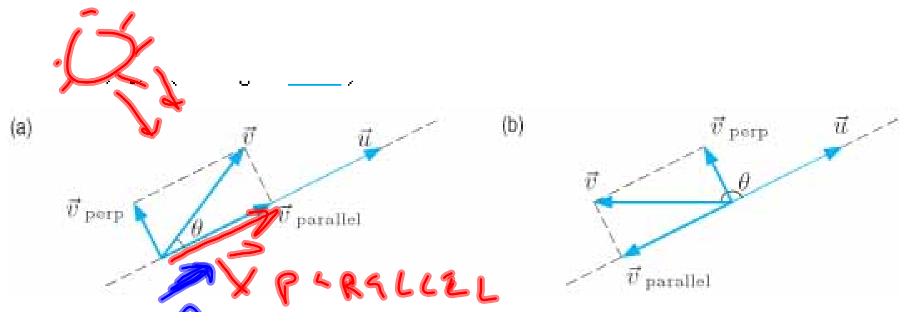



*instead of projecting a vector on the x-y plane,
we project it on another vector




$\vec{v} \parallel \vec{u}$

$\vec{x} \perp \vec{u}$




the projection of v on u, v parallel, measures the distance of v in the u direction.

it is the shadow on u.

v parallel is in the direction of u, therefore we will use the unit vector in the u direction

$$\frac{\vec{u}}{\|\vec{u}\|}$$

magnitude of $\vec{V}_{parallel}$

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

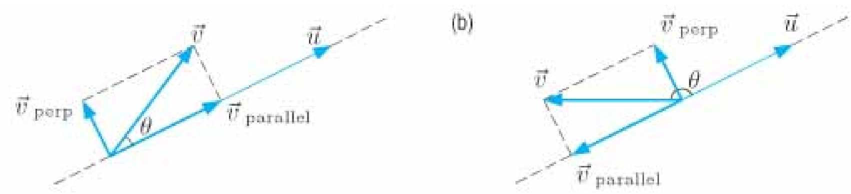
$$\vec{V}_{parallel} = \cancel{|\vec{u}|}|\vec{v}|\cos\theta \cdot \frac{\vec{u}}{\cancel{|\vec{u}|}}$$

$$= \|\vec{v}\|\cos\theta\,\vec{u}$$

Force u to be a unit vector.

$$\vec{v}_{\text{parallel}} = (\| \vec{v} \| \cos\theta) \vec{u} = (\vec{v} \cdot \vec{u}) \vec{u}.$$

What about v perpendicular??

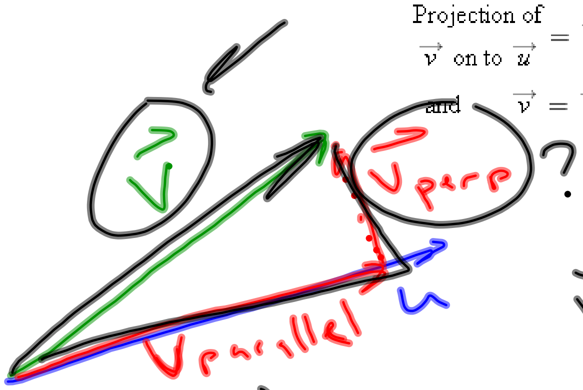


## Projection of $\vec{v}$ on the Line in the Direction of the Unit Vector $\vec{u}$

If $\vec{v}_{\text{parallel}}$ and $\vec{v}_{\text{perp}}$ are components of $\vec{v}$ which are parallel and perpendicular, respectively, to $\vec{u}$, then

$$\begin{array}{c}\text{Projection of}\\ \vec{v}\text{ on to }\vec{u}\end{array} = \vec{v}_{\text{parallel}} = (\vec{v}\cdot\vec{u})\,\vec{u} \qquad \text{provided } \|\vec{u}\| = 1$$

and $\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\text{perp}}$ so $\boxed{\vec{v}_{\text{perp}} = \vec{v} - \vec{v}_{\text{parallel}}}$.




$$V_{parallel} = (\vec{v}\cdot\vec{u})\,\vec{u}$$

$$\vec{V}_{parallel} + \vec{V}_{perp} = \vec{V}$$

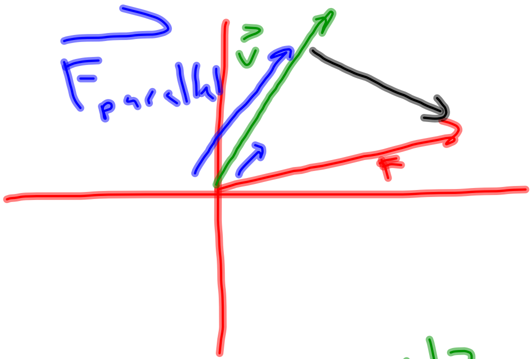
$$\vec{V}_{perp} = \vec{V} - \vec{V}_{parallel}$$

given vector $\vec{v} = 3\vec{i} + 4\vec{j}$ and force vector $\vec{F}$, find:

(a) The component of $\vec{F}$ parallel to $\vec{v}$.
(b) The component of $\vec{F}$ perpendicular to $\vec{v}$.

$$\vec{U_v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{3\vec{i} + 4\vec{j}}{5} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}$$

18. $\vec{F} = 4\vec{i} + \vec{j}$




$$F_{parallel} = \left\{\underline{(4\vec{i} + \vec{j}) \cdot \left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)}\right\} \times \left\{\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right\}$$

$$= \left(\frac{12}{5} + \frac{4}{5}\right) \cdot \left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)$$

$$= \left(\frac{16}{5}\right)\left(\frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}\right)$$

$$= \frac{48}{25}\vec{i} + \frac{64}{25}\vec{j}$$

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$$\vec{F}_{perp} = (4\vec{i} + \vec{j}) - \left(\frac{48}{25}\vec{i} + \frac{64}{25}\vec{j}\right)$$

$$= \left(4 - \frac{48}{25}\right)\vec{i} + \left(1 - \frac{64}{25}\right)\vec{j}$$

$\vec{a} = 2\vec{j} + \vec{k} \qquad \vec{b} = -3\vec{i} + 5\vec{j} + 4\vec{k} \qquad \vec{c} = \vec{i} + 6\vec{j}$

$\vec{y} = 4\vec{i} - 7\vec{j} \qquad \vec{z} = \vec{i} - 3\vec{j} - \vec{k}$

5. $\vec{c} \cdot \vec{a} + \vec{a} \cdot \vec{y}$
6. $\vec{a} \cdot (\vec{c} + \vec{y})$
7. $(\vec{a} \cdot \vec{b})\,\vec{a}$

In Exercises 10, 11, 12, 13 and 14, find a normal vector to the plane.

**10.** $2x + y - z = 23$

**11.** $1.5x + 3.2y + z = 0$

**12.** $z = 3x + 4y - 7$