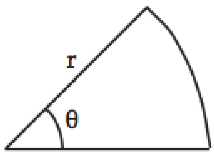
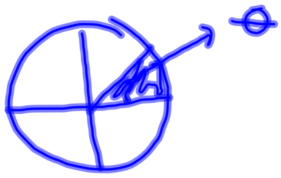
Area and Arc Length in Polar Coordinates

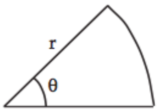




The area of the sector of a circle is $A = \frac{1}{2}\theta r^2$

$$\frac{\theta}{2\pi} = \frac{A_s}{\pi r^2}$$

$$A_s = \frac{1}{2}\theta r^2$$



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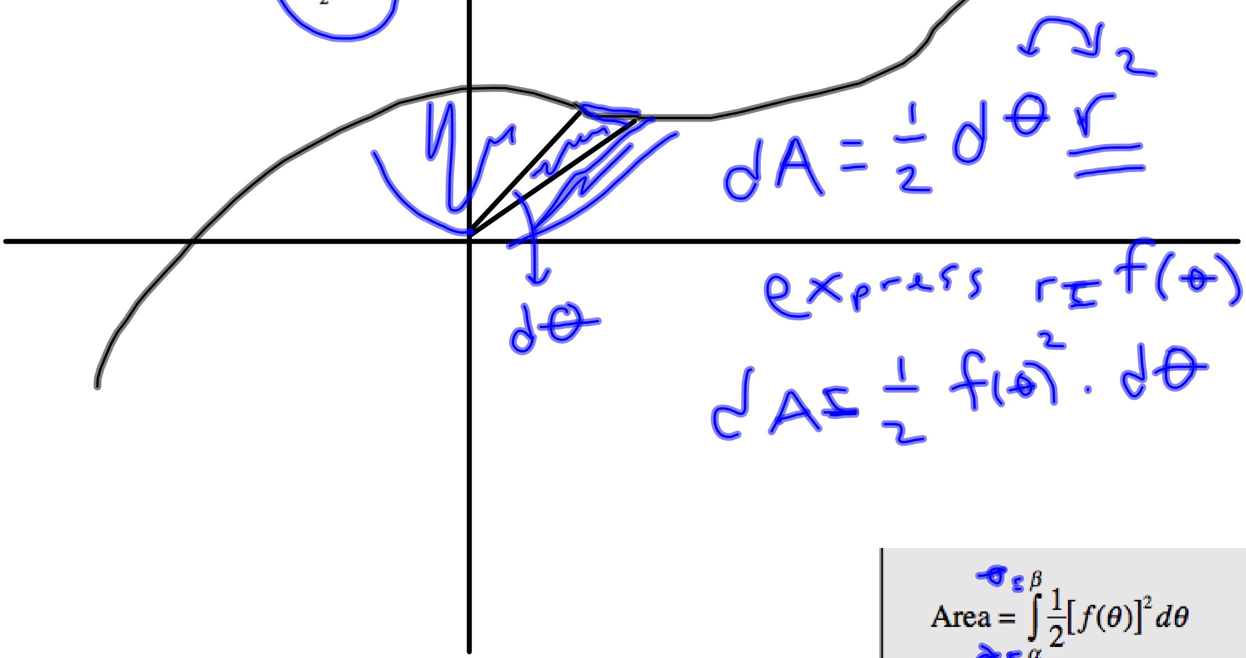


$$dA = \frac{1}{2}\, d\theta\, r^2$$

express $r = f(\theta)$

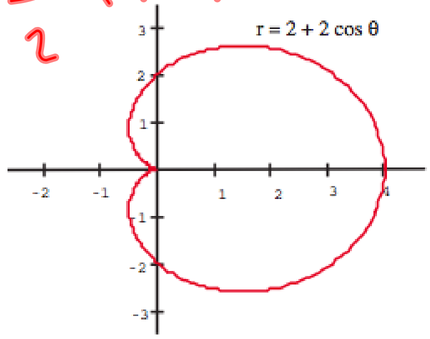
$$dA = \frac{1}{2} f(\theta)^2 \cdot d\theta$$

$$\text{Area} = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}[f(\theta)]^2\, d\theta$$

Find the area of the region bounded by the cardioid $r = 2 + 2\cos\theta$.

Using the fact that $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$.

$$dA = \frac{1}{2} f(\theta)^2 \, d\theta$$



$$\int_0^{2\pi} \frac{1}{2}(2+2\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (4 + 8\cos\theta + 4\cos^2\theta) \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4 + 8\cos\theta + 4\left(\frac{1}{2}(1+\cos(2\theta))\right) d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left[4 + 8\cos\theta + 2 + 2\cos(2\theta)\right] d\theta$$

$$= \frac{1}{2}\left[4\theta + 8\sin\theta + 2\theta + \sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[4(2\pi) + 2(2\pi)\right]$$

$$= 6\pi \text{ units}$$

Arc Length in Polar Coordinates

Remember the following:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta. \end{cases}$$

$$\begin{cases} x = f(\theta)\cos\theta \\ y = f(\theta)\sin\theta \end{cases}$$

$$L = \int_{\theta=\alpha}^{\theta=\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Now:
Use the chain rule to simplify L

$$\frac{dx}{d\theta} = f'(\theta)\cdot\cos\theta + f(\theta)(-\sin\theta)$$

$$\frac{dy}{d\theta} = f'(\theta)\cdot\sin\theta + f(\theta)\cos(\theta)$$

$$\frac{dx}{d\theta} = f'(\theta)\cdot\cos\theta + f(\theta)(-\sin\theta) \qquad L = \int \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dy}{d\theta} = f'(\theta)\cdot\sin\theta + f(\theta)\cos(\theta)$$

$$\left(\frac{dx}{d\theta}\right)^2 = (f'(\theta))^2\cdot\cos^2\theta + f(\theta)^2\sin^2\theta$$

$$+ \quad + \quad +$$

$$\left(\frac{dy}{d\theta}\right)^2 = (f'(\theta))^2\cdot\sin^2\theta + f(\theta)^2\cos^2\theta$$

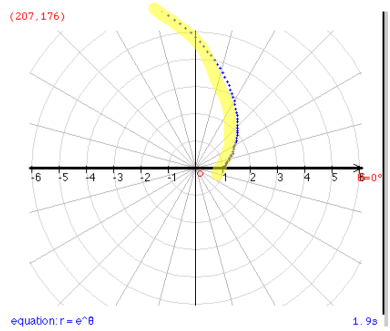
$$= f'(\theta)^2 + f(\theta)^2$$

$$= f'(\theta)^2 + f(\theta)^2$$

$$L = \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2}\, d\theta$$

$$L = \int_{\alpha}^{\beta} \sqrt{[(f'(\theta)]^2 + [(f(\theta)]^2}\ d\theta$$

Find the length of the logarithmic spiral $r = e^{\theta}$ on the interval $[0, 2\pi]$.



$f(\theta) = e^{\theta}$

$f'(\theta) = e^{\theta}$

$$L = \int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\ d\theta$$

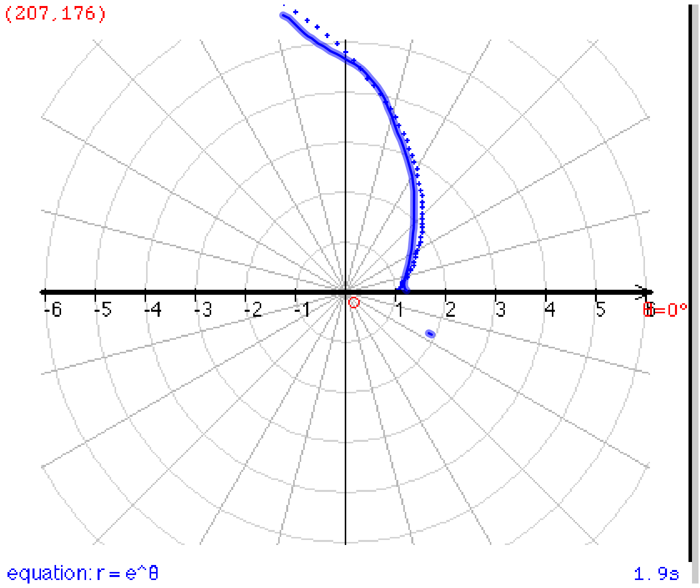
$$= \int_0^{2\pi} \sqrt{2 \cdot (e^{\theta})^2}\ d\theta$$

$$= \int_0^{2\pi} \sqrt{2} \cdot \sqrt{(e^{\theta})^2}\ d\theta$$

$$= \sqrt{2} \int_0^{2\pi} e^{\theta}\ d\theta$$

$$= \sqrt{2} \cdot e^{\theta} \Big|_0^{2\pi}$$

$$\boxed{= \sqrt{2}\left[e^{2\pi} - e^{0}\right]}$$


(207,176)
-6
-5
-4
-3
-2
-1
1
2
3
4
5
6
θ=0°
equation: r = e^θ
1.9s

Polar Coordinates

(θ, r)

Area = $\int \frac{1}{2} f(\theta)^2 \, d\theta$

Arclength = $\int \sqrt{f'(\theta)^2 + f(\theta)^2} \, d\theta$