Limits and Differentiability of Vector-Valued Functions

$$S(t) = f(x)\,i + g(t)\,j$$

Limit of a Vector-valued Function

If $\boldsymbol{R}$ is a vector-valued function such that $\boldsymbol{R}(t) = f(t)\cdot\boldsymbol{i} \oplus g(t)\cdot\boldsymbol{j}$, then

$$\lim_{t\to a}\boldsymbol{R}(t) = [\lim_{t\to a} f(t)]\cdot\boldsymbol{i} \oplus [\lim_{t\to a} g(t)]\cdot\boldsymbol{j}$$

provided that f and g have limits at $t = a$.

If $\boldsymbol{R}$ is defined by $\boldsymbol{R}(t) = \cos t\cdot\boldsymbol{i} \oplus \sin t\cdot\boldsymbol{j}$, find $\lim_{t\to\pi/2}\boldsymbol{R}(t)$

$$= \cos\left(\tfrac{\pi}{2}\right)i \oplus \sin\left(\tfrac{\pi}{2}\right)j$$

$$= 1j$$

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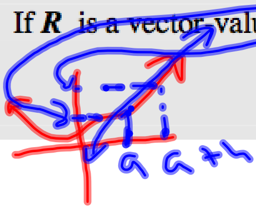
Derivative of a Vector-valued Function

If $\boldsymbol{R}$ is a vector-valued function, its derivative $\boldsymbol{R}'$ at the input a is defined by



$$\boldsymbol{R}'(a) = \lim_{h\to 0} \boxed{\frac{\boldsymbol{R}(a+h) \ominus \boldsymbol{R}(a)}{h}}$$

R = fi + gj

$$\boldsymbol{R}'(a) = \lim_{h\to 0} \frac{\boldsymbol{R}(a+h) \ominus \boldsymbol{R}(a)}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)\cdot\boldsymbol{i} \oplus g(a+h)\cdot\boldsymbol{j} \ominus f(a)\cdot\boldsymbol{i} \ominus g(a)\cdot\boldsymbol{j}}{h}$$

$$= \lim_{h\to 0} \left\{ \left[\frac{f(a+h)-f(a)}{h}\right]\boldsymbol{i} \oplus \left[\frac{g(a+h)-g(a)}{h}\right]\cdot\boldsymbol{j} \right\}$$

$$= \left\{ \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h}\right] \right\}\boldsymbol{i} \oplus \left\{ \lim_{h\to 0} \left[\frac{g(a+h)-g(a)}{h}\right] \right\}\cdot\boldsymbol{j}$$

$$= f'(a)\cdot\boldsymbol{i} \oplus g'(a)\cdot\boldsymbol{j}$$

If $\boldsymbol{R}(t) = \cos t \cdot \boldsymbol{i} \ \oplus \sin t \cdot \boldsymbol{j}$, find $\boldsymbol{R}'(t)$.

$R'(t) = -\sin(t)\,i \oplus \cos(t)\,j$

Velocity Vector and Acceleration Vector

If $\boldsymbol{R}$ is a vector-valued function, $\boldsymbol{R}'$ is called the **velocity vector** and $\boldsymbol{R}''$ is called the **acceleration vector**.

$\cos^{-1}(t)$

A particle travels in the plane so that its position at time t is given by

$\boldsymbol{R}(t) = t \cdot \boldsymbol{i} \oplus \ln(\sec t)\cdot \boldsymbol{j}, 0 \le t \le \frac{\pi}{2}$. Determine the speed of the particle at $t = \frac{\pi}{4}$.

$$R'(t) = 1\,i \oplus \frac{1}{\sec(t)}\cdot(-1)\cdot\cos^{-2}(t)(-\sin t))\,j$$

$$R'\left(\frac{\pi}{4}\right) = 1\,i \oplus \cos\left(\frac{\pi}{4}\right)\cdot\cos^{-2}\left(\frac{\pi}{4}\right)\cdot\sin\left(\frac{\pi}{4}\right)j$$

$$= 1\,i \oplus \frac{\sqrt{2}/2\cdot\sqrt{2}/2}{(\sqrt{2}/2)^2}\,j$$

$$= 1\,i \oplus j$$

$$\text{Speed} = \sqrt{R'_x(t) + R'_y(t)}$$

$$= \sqrt{1^2 + 1^2}$$

$$= \sqrt{2}$$

The acceleration vector of a particle moving in a plane is given by the vector-valued function $\boldsymbol{R}''(t) = 12t \cdot \boldsymbol{i} \oplus 2 \cdot \boldsymbol{j}$. When $t = 0$, the particle is at the point $(0, -1)$ and its velocity is $-\boldsymbol{i}$. Find its position at time $t = 2$.

$$V = \int a \, dt$$

$$= \int (12t\,i \oplus 2j)\,dt$$

$$R' = (6t^2 + c_x)\vec{i} \oplus (2t + c_y)\vec{j}$$

$$t = 0$$

$$(6(0)^2 + c_x)\vec{i} \oplus (2 \cdot 0 + c_y)\vec{j} = -\vec{i}$$

$$c_x \vec{i} \oplus c_y \vec{j} = -\vec{i}$$

$$c_x \vec{i} = -\vec{i} \qquad c_y = 0$$

$$c_x = -1$$

$$V = R' = (6t^2 - 1)\vec{i} \oplus (2t + 0)\vec{j}$$

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$$V = R' = (6t^2 - 1)\,\mathring{i} \oplus (2t + 0)\,\mathring{j}$$

$$R = \int [(6t^2 - 1)\,\mathring{i} \oplus (2t)\,\mathring{j}]\,dt$$

$$= (2t^3 - t + C_x)\,\mathring{i} \oplus (t^2 + C_y)\,\mathring{j}$$

$$= C_x\,\mathring{i} \oplus C_y\,\mathring{j} = 0\,\mathring{i} - \mathring{j}$$

$$C_x = 0, \quad C_y = -1.$$

$$R(t) = (2t^3 - t + 0)\,\mathring{i} \oplus (t^2 - 1)\,\mathring{j}$$

$$R(2) = (2 \cdot 2^3 - 2)\,\mathring{i} \oplus (2^2 - 1)\,\mathring{j}$$

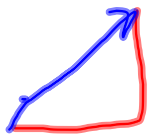
$$= 14\,\mathring{i} \oplus 3\,\mathring{j}$$

If the position vector for a moving particle is determined by $\boldsymbol{R}(t) = \frac{1}{2}t^2 \cdot \boldsymbol{i} \oplus t \cdot \boldsymbol{j}$, find the velocity and acceleration vectors at $t = 2$. Sketch arrows that represent $\boldsymbol{R}(2)$, $\boldsymbol{R}'(2)$, and $\boldsymbol{R}''(2)$.

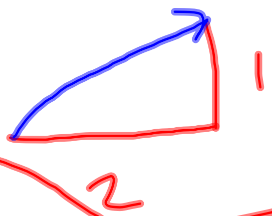
$$v = R'(t) = t\,i \oplus j$$

$$a = R''(t) = i$$

$$R(2) = \frac{1}{2} \cdot 2^2\, i \oplus 2j = 2i + 2j$$



$$R'(2) = 2i + j$$



$$R''(2) = i$$