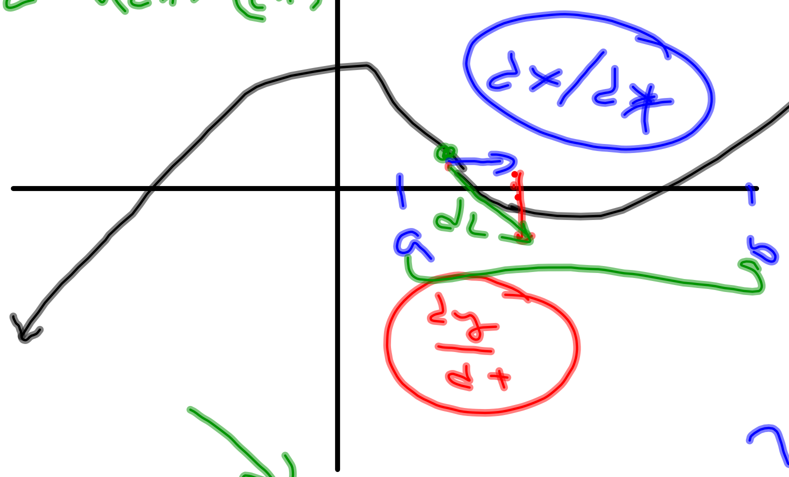
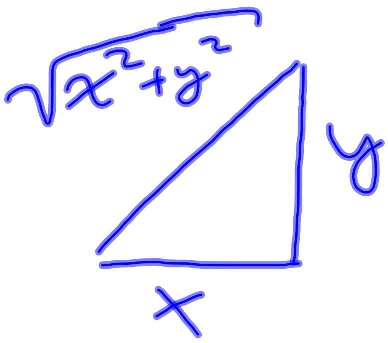
Length of an Arc Described by Parametric Equations

$dL = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\frac{dx}{dt} + \frac{dy}{dt} =$

Arc length





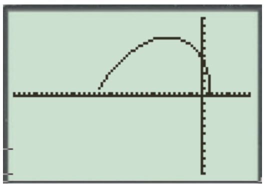
$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Find the length of the curve C defined for $0 \le t \le \pi$ by

$$\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$$

$$\frac{dx}{dt} = e^t(-\sin(t)) + e^t(\cos(t))$$



$$\frac{dy}{dt} = e^t(\cos(t)) + e^t \sin(t)$$

$$\int_0^{\pi} \sqrt{\left(e^t(-\sin t) + e^t \cos t\right)^2 + \left(e^t \cos t + e^t \sin t\right)^2}$$

Solution:

We have $\dfrac{dx}{dt} = e^t \cos t - e^t \sin t = e^t(\cos t - \sin t)$

$\dfrac{dy}{dt} = e^t(\sin t) + e^t \cos t = e^t(\cos t + \sin t).$

Then $\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2 = e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2$

$= e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t)$

$= 2e^{2t}.$

Thus, we obtain

$$\int_0^{\pi} \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_0^{\pi} \sqrt{2e^{2t}}\, dt = \int_0^{\pi} \sqrt{2}\, e^t dt = \sqrt{2}\left[e^t\right]_0^{\pi} = \sqrt{2}(e^{\pi} - 1).$$

$$\int_0^{\pi}\sqrt{\left(e^t(-\sin t)+e^t\cos t\right)^2+\left(e^t\cos t+e^t\sin t\right)^2}$$

$$=\int_0^{\pi}\sqrt{e^{2t}\sin^2 t+e^{2t}\cos^2 t-2e^{2t}\cancel{\sin t\cos t}+e^{2t}\cos^2 t+e^{2t}\sin^2 t+2e^{2t}\cancel{\cos t\sin t}}\,dt$$

$$=\int_0^{\pi}\sqrt{e^{2t}\left(\sin^2 t+\cos^2 t+\cos^2 t+\sin^2 t\right)}\,dt$$

$$=\int_0^{\pi}\sqrt{2\cdot e^{2t}}\,dt$$

$$=\int_0^{\pi}\sqrt{2}\cdot e^{t}\,dt$$

$$=\sqrt{2}\,e^{t}\Big|_0^{\pi}$$

$$=\sqrt{2}\left[e^{\pi}-1\right]$$

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$$= e^{2t}(\cos^2 t - 2\cos t\,\sin t + \sin^2 t + \cos^2 t + 2\cos t\,\sin t + \sin^2 t)$$

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Thus, we obtain

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In Exercises 1–7, find the length of the arc described.

1. $\begin{cases} x = t^2 \\ y = t^3 \end{cases}$ $0 \le t \le 1$

2.

$$\frac{dx}{dt} = 2t$$

$$\frac{dy}{dt} = 3t^2$$

$$L = \int_0^1 \sqrt{(2t)^2 + (3t^2)^2}\, dt$$

$$L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt$$

$$L = \int_0^1 t\sqrt{4 + 9t^2}\, dt$$

$$= 1.44$$

1. At time t, a particle moving in the xy-plane is at position $(x(t), y(t))$, where $x(t)$ and $y(t)$ are not explicitly given. For $t \geq 0$, $\frac{dx}{dt} = 4t + 1$ and $\frac{dy}{dt} = \sin(t^2)$. At time $t = 0$, $x(0) = 0$ and $y(0) = -4$.

(a) Find the speed of the particle at time $t = 3$, and find the acceleration vector of the particle at time $t = 3$.

(b) Find the slope of the line tangent to the path of the particle at time $t = 3$.

(c) Find the position of the particle at time $t = 3$.

~~(d) Find the total distance traveled by the particle over the time interval~~ $0 \leq t \leq 3$.

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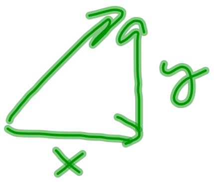
$\frac{dx}{dt} = 4(3)+1 = 13$ m/s in x

(a) $\frac{dy}{dt} = \sin(3^2) = \sin(9)$ in y



$v = \sqrt{13^2 + \sin^2(9)}$

acceleration $a_x = \frac{d^2x}{dt^2} = 4$

$a_y = \frac{d^2y}{dt^2} = \cos(t^2) \cdot 2t$

$a_{xy} = \sqrt{4^2 + \cos(9) \cdot 6}$

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$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin(t^2)}{4t+1}$$

$$= \frac{\sin(9)}{13}$$

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$$\int_0^3 \frac{dx}{dt}\,dt = x(3) - x(0) = 0$$

$$\int_0^3 (4t+1)\,dt = 2t^2 + t\Big|_0^3$$

$$= 2 \cdot 9 + 3$$

$$= 21 = x(3) - 0$$

$$x(3) = 21$$

y: $y(3) = \int_0^3 \sin(t^2)\,dt + y(0)$

$$= .77 + -4$$

$$y(3) = -3.23$$

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total distance = Arclength

$$\int_0^3 \sqrt{(4t+1)^2 + (\sin(t^2))^2}\, dt$$

$= 21.1$ units