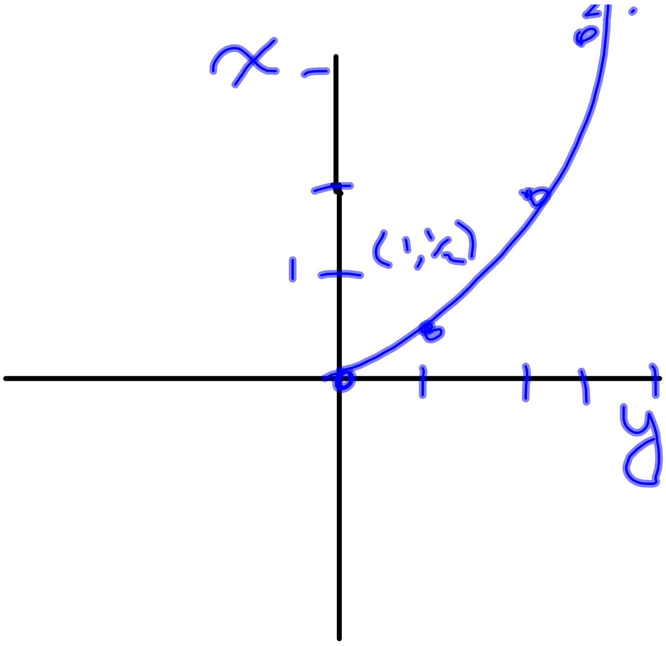
Parametric Equations

$$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$$

Example: $\boxed{\begin{array}{l} x=t \\ y=1/2*t^2 \end{array}}$

t	0	1	2	3	4
x	0	1	2	3	4
y	0	$\frac{1}{2}$	2	4.5	8



When the point (x, y) is described by two coordinate functions f and g so that x is the value f relates to t and y is the value g relates to t, then we say that we have a **parametrization** of the points (x, y) in terms of the **parameter** t. A collection of points (x, y) defined in this way is called a **plane curve**.

Convert between parametric equation and standard rectangular ~~equaution~~ equation

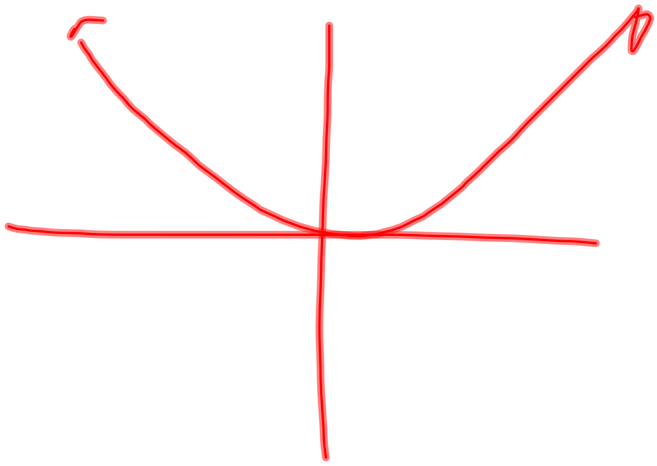
$x=t$

$y=1/2*t^2$

1. Solve x for t
2. substitute

$$\boxed{y(x) = \frac{1}{2}x^2}$$

$\frac{1}{2}x^2$



$$\boxed{x(t) = 3t - 1}$$

$$y(t) = e^{t}$$

$$x + 1 = 3t$$

$$\boxed{t = \frac{x+1}{3}}$$

$$t(x) = \frac{x+1}{3}$$

$$y(x) = e^{\frac{x+1}{3}}$$

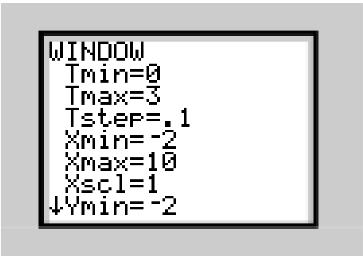
$y(t)$

$y(t(x))$

$y(x)$...

Parametric Mode on the Calculator

x=t
y=1/2*t^2



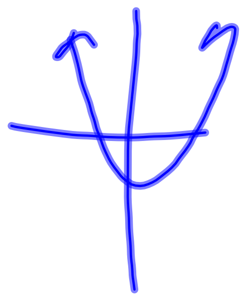
Graphing a parametric equation and Eliminating the parameter.

Try:

1. Graph: $x=\cos(t)$
$y=\cos(2t)$

2. Eliminate the parameter. Write in as a rectangular equation.
Hint: $\cos(2t)=2\cos^2(t)-1$

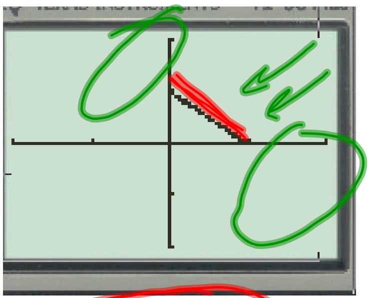
$$y = 2\cdot\cos^2(t) - 1$$
$$= 2(\cos(t))^2 - 1$$
$$\boxed{y(x) = 2\cdot x^2 - 1}$$



Sketch the curve defined by the parametric equations

$$\begin{cases} x = \cos^2\theta \\ y = \sin^2\theta . \end{cases}$$

$\theta \to T$



$y(x)$ not $y(\theta)$

$x(\theta)$

$x = \cos^2\theta$ ✓

$y = \sin^2\theta$ ✓

$x: [0,1]$

$y: [0,1]$

$x + y = \cos^2\theta + \sin^2\theta$

$x + y = 1$

$\boxed{y = -x + 1}$

$\sin\theta$

$[-1,1]$

$\cos\theta$

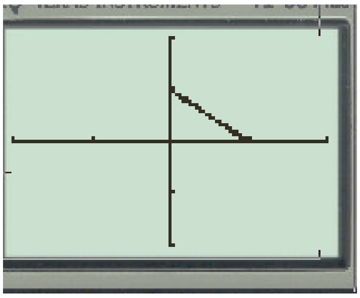
$[-1,1]$

$\sin^2\theta$

$y: [0,1]$

$\cos^2\theta$

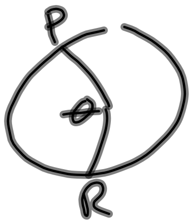
$x: [0,1]$



Example 5: *Parametric Equations for a Cycloid*

$\theta - \pi/2$

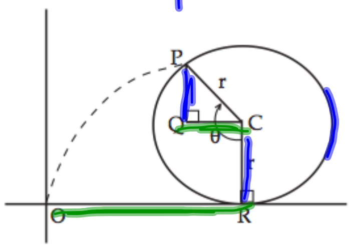
A circle of radius r rolls along the x-axis. Find parametric equations of the path traced out by the point of the circle that starts at the origin.



Solution:

We take the angle θ through which the circle has rolled at a given moment as the parameter. Then the coordinates of the point P are given by

$$\begin{cases} x = OR - QC \\ y = RC + PQ \end{cases}$$



Since the circle is rolling, the length of the segment $\overline{OR}$ is the same as the length of the arc PR,.

$PR = \frac{\theta}{2\pi}\cdot 2\pi r = \theta r$, where θ is the measure of $\angle RCP$.

rolled

$OR = PR$
$= \theta r$

Since $\angle QCP = \theta - \frac{\pi}{2}$, we have $QC = r\cos(\theta - \frac{\pi}{2})$ and $PQ = r\sin(\theta - \frac{\pi}{2})$.

Hence $\begin{cases} x = r\theta - r\cos(\theta - \frac{\pi}{2}) \\ y = r + r\sin(\theta - \frac{\pi}{2}) \end{cases}$

$\cos(\theta - \frac{\pi}{2}) = \sin\theta$
$\sin(\theta - \frac{\pi}{2}) = -\cos\theta$

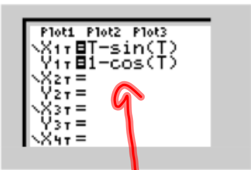
Simplifying, we obtain

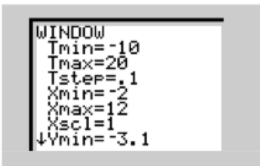
$$\begin{cases} x = r(\theta - \sin\theta) \\ y = r(1 - \cos\theta) \end{cases}$$

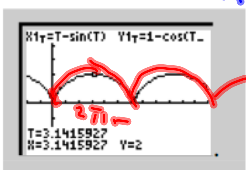
$r = 1$

$QC = r\cdot\cos(\theta - \pi/2)$
$QP = r\cdot\sin(\theta - \pi/2)$
$RC = r$

The curve described by these equations is called a **cycloid**.







The derivative in parametric equations:

(dy/dt)/(dx/dt) = ??

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{\cancel{dt}} \cdot \frac{\cancel{dt}}{dx} = \frac{dy}{dx}$$

$x = f(t)$ and $y = g(t)$.

$$\frac{dx}{dt} = f' \qquad \frac{dy}{dt} = g'$$

in terms of f(t) and g(t), find dy/dx.

$$\frac{dy}{dx} = \frac{g'}{f'}$$

given: x=cos(t) $x' = -\sin(t)$

y=sin(t) $y' = \cos(t)$

find: dy/dx for any "t"

$$\frac{dy}{dx} = \frac{\cos(t)}{-\sin(t)}$$

$$\frac{dy}{dx} = -\cot(t)$$

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Find the points at which $\frac{dy}{dx}$ does not exist for the curve defined parametrically by $x = \sec\theta$ and $y = \tan\theta$.

$$\frac{d}{d\theta}\left[\cos(\theta)\right]^{-1}$$

$$= -1 \cdot \cos^{-2}\theta\,(-\sin\theta)$$

$$\frac{dx}{d\theta} = \frac{\sin\theta}{\cos^2\theta} \qquad \theta \neq \frac{\pi}{2} + k\pi$$

$$\frac{d}{d\theta}\tan\theta$$

$$= \sec^2\theta$$

$$\frac{dy}{d\theta} = \sec^2\theta = 1/\cos^2\theta \qquad \theta \neq \frac{\pi}{2} + k\pi$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{1/\cancel{\cos^2\theta}}{\sin\theta/\cancel{\cos^2\theta}}$$

$$\frac{dy}{dx} = \frac{1}{\sin\theta} \qquad \theta \neq 0 + k\pi$$

$$\theta \neq 0 + k\pi,\ \frac{\pi}{2} + k\pi$$

$$\boxed{\theta \neq 0 + \frac{k\cdot\pi}{2}}$$

Find the ~~slope of the~~ tangent to the cycloid line @ $\theta = \frac{\pi}{2}$.

$$\begin{cases} x = \theta - \sin\theta \\ y = 1 - \cos\theta \end{cases}$$

$x\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)$

$= \frac{\pi}{2} - 1$

$y\left(\frac{\pi}{2}\right) = 1 - \cos\left(\frac{\pi}{2}\right) = 1$

$x' = 1 - \cos\theta$

$y' = +\sin\theta$

$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{\sin\left(\frac{\pi}{2}\right)}{1-\cos\left(\frac{\pi}{2}\right)}$

$= 1$

$$\boxed{y = 1\left(x - \left(\frac{\pi}{2} - 1\right)\right) + 1}$$

The Second Derivative!!

$$\frac{d^2y}{dx^2}=\frac{dy'}{dx}=\frac{\frac{dy'}{dt}}{\frac{dx}{dt}}=\frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$$

$$\frac{d^2y}{dx^2}=\frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dx}=\boxed{\frac{\frac{dy}{dt}}{\frac{dx}{dt}}}$$

$$\frac{d^2y}{dx^2}\quad\frac{\frac{d\left(\frac{dy}{dx}\right)}{dt}}{\frac{dx}{dt}}$$

Find the slope of the tangent to the cycloid

$$\begin{cases} x = \theta - \sin\theta \\ y = 1 - \cos\theta \end{cases}$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d\left(\frac{dy}{dx}\right)}{dt}}{\dfrac{dx}{dt}}$$

DONE! Now, find the second derivative of the cycloid:

$x' = 1 - \cos\theta$

$y' = \underline{\underline{\sin\theta}}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}(\sin\theta)}{dx/d\theta} = \boxed{\frac{\cos\theta}{1-\cos\theta}}$$