Convergence

$1, + 3, + 5, + 7, + 9, + 11 \ldots$

$$f(n) = a_n = \sum_{n=1}^{\infty} 2n - 1$$

$$\begin{aligned} &(2(1)-1) = 1 \\ &(2(2)-1) = 3 \\ &(2(3)-1) = 5 \end{aligned}$$

2,4,6,8,10...

$$\sum_{n=1}^{\infty} 2n$$

From before, we learned how to write sin(x), cos(x), e^x as:

$f(x) \approx P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

The general form for all power series.

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

$x = 0$

is called a **power series centered at $x = 0$**. More generally, a series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots$$

is called a **power series centered at c**, where c is a constant.

$x = c$

$$\sum_{n=0}^{\infty} a_n x^n$$

For what values of x does the series converge or diverge?

$= a_0 + a_1 x^1 + a_2 x^2 \ldots\ldots$

$|r| < 1 \quad , \quad r = x$

$|x| < 1$

Converges: $-1 < x < 1$

Converge to: $\dfrac{a_0}{1-r} = \boxed{\dfrac{a_0}{1-x}}$

For what x's does the power series

$$\sum_{n=0}^{\infty} x^n = \boxed{1} + x + x^2 + x^3 + \cdots$$

converge, and what is its sum?

$$\sum 1 \cdot (x)^n$$

$$|x| < 1$$

for $\underline{-1 < x < 1}$, converge to: $\boxed{\dfrac{1}{1-x}}$

The Ratio Test

Let $\sum a_n$ be a series of nonzero terms, and suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$.

$a_{n+1} < a_n$

Then,

Converge

a) the series converges if $L < 1$,

b) the series diverges if $L > 1$, $a_{n+1} > a_n$ $a_{n+1} < a_n$

c) the test is inconclusive if $L = 1$.

$a_{n+1} = a_n$

$$\lim_{n\to\infty} \boxed{\frac{|a_{n+1}|}{|a_n|}} = L$$

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \stackrel{*}{=} \frac{(n+1)!\cdot x^{n+1}}{n!\cdot x^n}$$

$$\stackrel{*}{=} \frac{(n+1)\cancel{n!}\cdot x^1 \cdot \cancel{x^n}}{\cancel{n!}\,\cancel{x^n}}$$

$$\lim_{n\to\infty} \left|(n+1)\cdot x\right| < 1$$

if $\boxed{x=0}$

$$\lim_{n\to\infty} |(n+1)\cdot x| < 1$$

converges

Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots$$

which is the Maclaurin series for $f(x)=e^x$.

$$\boxed{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} < 1$$

$$\lim_{n\to\infty}\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}=\lim_{n\to\infty}\frac{\frac{x\cdot\cancel{x^n}}{(n+1)\cancel{n!}}}{\cancel{\frac{x^n}{n!}}}$$

$$=\lim_{n\to\infty}\left|\frac{x}{n+1}\right|=0<1$$

converges for all $\underline{\underline{x}}$

Find the interval of convergence for

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)2^n} = 1 + \frac{1}{2}\cdot\frac{x}{2} + \frac{1}{3}\cdot\frac{x^2}{2^2} + \frac{1}{4}\cdot\frac{x^3}{2^3} + \cdots$$

$$\lim_{n\to\infty} \left| \frac{\dfrac{x^{n+1}}{(n+1+1)2^{n+1}}}{\dfrac{x^n}{(n+1)2^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x \cdot \cancel{x^n}}{(n+2)\cancel{2^n} \cdot 2} \cdot \frac{(n+1)\cancel{2^n}}{\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x(n+1)}{2(n+2)} \right|$$

$$= \lim_{n\to\infty} \left(\frac{xn+x}{2n+2} \right)$$

$$= \lim_{n\to\infty} \left| \frac{x}{2} \right| < 1$$

$$\left| \frac{x}{2} \right| < 1$$

$$-2 < x < 2$$

For a given power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ there are only three possibilities:

(i) The series converges for all x.

(ii) The series converges only when $x = c$.

(iii) There is a positive number R such that the series converges if $|x-c| < R$ and diverges if $|x-c| > R$. The series may or may not converge at either of the endpoints $x = c - R$ and $x = c + R$.

Find the radius of convergence of the series $\sum_{n=0}^{\infty} 2(x-3)^n$

$$\lim_{n\to\infty} \left| \frac{2(x-3)^{(n+1)}}{2(x-3)^n} \right|$$

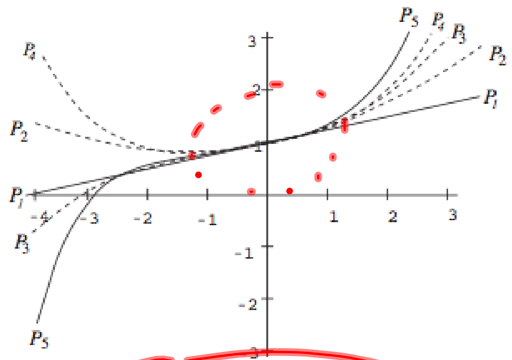
$$= \lim_{n\to\infty} \left| \frac{\cancel{2}(x-3)\cancel{(x-3)^n}}{\cancel{2(x-3)^n}} \right|$$

$$= \lim_{n\to\infty} |x-3| = |x-3| < 1$$

$$\boxed{2 < x < 4}$$

Important Theory: $f(x) \approx P_n(x)$

The radius of convergence for a polynomial function gives the domain over which a power series approximates the original function.



$P_n(x)$ works!!

$-1 < x < 1$