11.1 Polynomial Approximations of Functions

How do we get the polynomial?

-approximate any function as a polynomial (for now only when x=0)

2nd degree

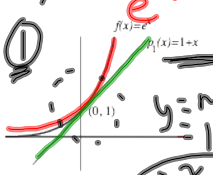
Tangent

$f(0) = e^0 = 1$

$f'(0) = e^0 = 1$

e^x

①

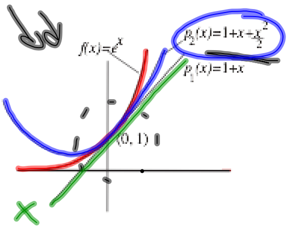


$y = m(x - x_1) + y_1$

$= 1(x - 0) + 1$

$\boxed{= x + 1}$

②



$y = e^x$

The important thing is that each polynomial approximation must have the same derivative(s) as the function and contain the same point.

$$f(x) \approx P_n(x)$$

"polynomial function of... degree 'n'"

$$f(0) = P_n(0)$$

$$e^0 = \boxed{1 = P_n(0)}$$

$$f(x) = e^x$$

$$f'(x) = e^x$$

$$f'(0) = P_n'(0)$$

$$\boxed{1 = P_n'(0)}$$

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$P_n(0) = \boxed{a_0 = 1}$$

$$P_n'(x) = a_1 + \cancel{2a_2 x} + \cancel{3a_2 x^2} \ldots$$

$$\boxed{1 = a_1}$$

TAYLOR POLYNOMIAL AT X=0.

$$f(x) \approx p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

For the value of the polynomial $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ to agree with the value of f at $x = 0$, we must set $a_0 = f(0)$.

The choice of the coefficient a_1 is based on our desire that $f'(0) = p_n'(0)$. Differentiating $p_n(x)$ yields

$$p_n'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1},$$ $\boxed{x=0}$

and substituting $x = 0$ shows that $p_n'(0) = a_1$, so we choose $a_1 = f'(0)$.

$$P_n'(x) = a_1 + \cancel{2a_2 x} + \cancel{3a_3 x^2} + \ldots$$

$$P_n''(x) = 2a_2 + \cancel{3 \cdot 2 \cdot a_3 x} + \ldots$$

$$P_n'''(x) = 3 \cdot 2 \cdot a_3 + \ldots$$

$$f(0) = P_n(0) \rightarrow \cancel{P_n(0)}\, f(0) = a_0$$

$$f'(0) = P_n'(0) \quad f'(0) = a_1$$

$$f''(0) = P_n''(0) \quad f''(0) = 2a_2$$

$$\frac{f''(0)}{2 \cdot 1} = a_2$$

$$f'''(0) = P_n'''(0) \quad f'''(0) = 3 \cdot 2 a_3$$

$$a_3 = \frac{f'''(0)}{3 \cdot 2 \cdot 1}$$

If the function f has n derivatives at $x = 0$, then the polynomial

$$\boxed{p_n(x)} = \boxed{f(0) + f'(0)x} + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$2 \cdot 1$ $\quad 3 \cdot 2 \cdot 1$

is called a **Taylor polynomial for f at $x = 0$.**

$f(x) = e^x$

Find Taylor Polynomials p_1, p_2, and p_3 for $y=e^X$.

$f(0) = e^0 = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$f'''(x) = e^x$ $\quad f'''(0) = 1$

$P_1 = 1 + 1x$

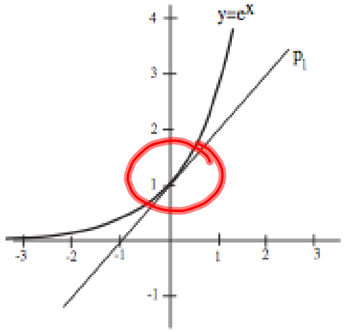
$P_2 = 1 + 1x + \frac{1}{2 \cdot 1} \cdot x^2$

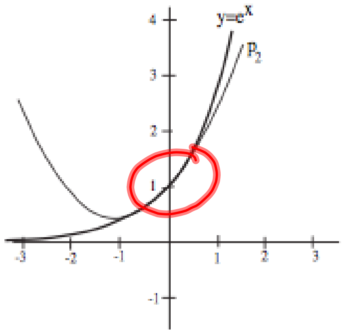
$P_3 = 1 + X + \frac{1}{2}X^2 + \frac{1}{3 \cdot 2 \cdot 1} \cdot x^3$

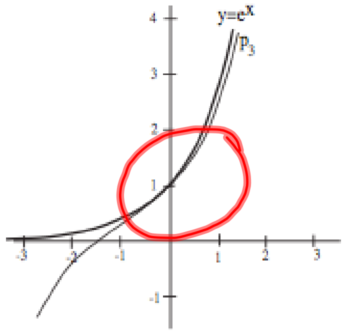
as $n\uparrow$, $P_n(x)$ improve

around $x=0$

P_1 P_2 P_3







Find Taylor (or Maclaurin) Polynomials for y=sin(x).

$$P_n(x) = \boxed{f(0)} + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$f(0) = \sin(0) = 0$

$f'(0) = \cos(0) = 1$

$f''(0) = -\sin(0) = 0$

$f'''(0) = -\cos(0) = -1$

$f^{(4)}(0) = \sin(0) = 0$

$f^{(5)}(0) = \cos(0) = 1$

$f^{(6)}(0) = 0$

$f^{(7)}(0) = -1$

$\boxed{n=1}$

$P_1 = 0 + 1x$

$P_2 = 0 + 1x + \frac{0}{2!}x^2$

$5 \cdot 4 \cdot 3 \cdot 2$

$$P_3 = 0 + 1x + 0 + \frac{-1}{3!} \cdot x^3$$

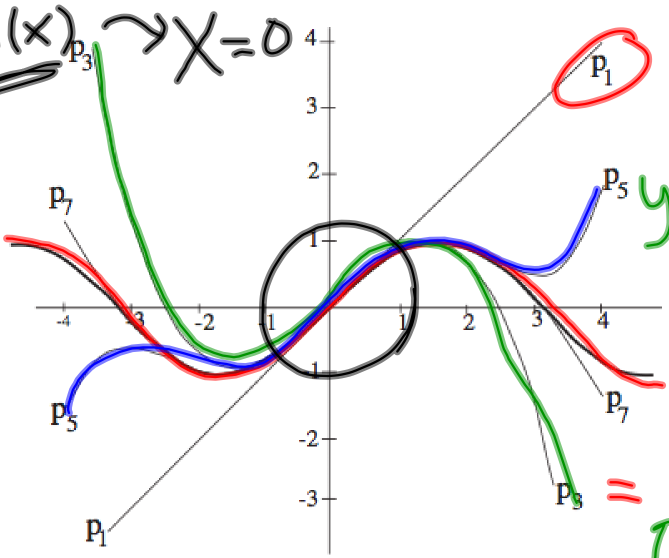
$$= 1x - \frac{1}{6}x^3$$

$$\cancel{P_4} = 1x - \frac{1}{6}x^3 + \cancel{\frac{f^{(4)}(0)}{4!}x^4}$$

$$P_5 = 1x - \frac{1}{6}x^3 + \frac{f^{(5)}(0)}{5!}x^5$$

$$= 1x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

$\sin(x) \to x=0$



$y = x$

$y = \frac{1}{120}x^5 - \frac{1}{6}x^3 + x$

$= -\frac{1}{6}x^3 + x$

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3\cdot 2} + \dots$$

Find Taylor (or Maclaurin) Polynomials for y=cos(x).

$x=0$

$f(x) = \cos(x)$

$f(0) = 1$

$f'(x) = -\sin(x)$	$f'(0) = \sin(0) = 0$
$f''(x) = -\cos(x)$	$f''(0) = -\cos(0) = -1$
$f'''(x) = \sin(x)$	$f'''(0) = 0$
$f^{(4)}(x) = \cos(x)$	$f^{(4)}(0) = 1$
$f^{(5)}(x) = -\sin(x)$	$f^{(5)}(0) = 0$
$f^{(6)}(x) = -\cos(x)$	$f^{(6)}(0) = -1$

$\cancel{P_5}$

$$P_0(x) = 1$$

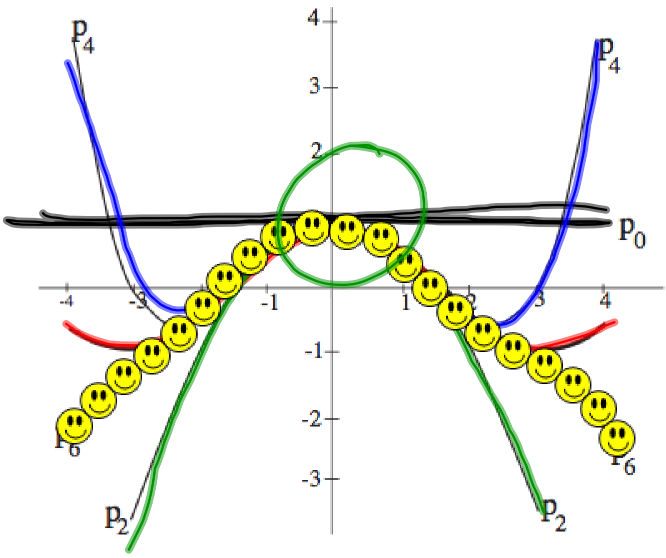
$$\cancel{P_1(x)} = 1 + \cancel{0x}$$

$$P_2(x) = 1 + \frac{-1}{2}x^2$$
$$= 1 - \frac{1}{2}x^2$$

$$P_4(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4$$

$$P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \frac{-1}{6!}x^6$$

$$\boxed{P_6(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6}$$



Approximate

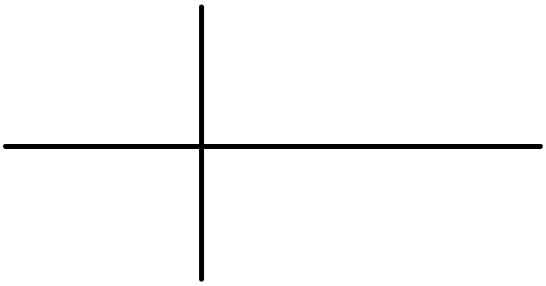
$y = \cos x$

for x

close to

$x = 0$

Taylor Polynomials around the point x=c



If the function f has n derivatives at $x = c$, then the polynomial

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots + \frac{f^{(n)}(c)(x-c)^n}{n!}$$

is called the **Taylor polynomial for f at $x = c$**.

Find Taylor Approximations for y=ln(x) at x=1.

P_1, P_2, P_3

$$P_n(x) = f(1) + f'(1)\cdot(x-1) + \frac{f''(1)}{2!}\cdot(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 \ldots$$

$f(1) = \ln(1) = 0$

$f'(x) = \frac{1}{x}$, $f'(1) = 1$

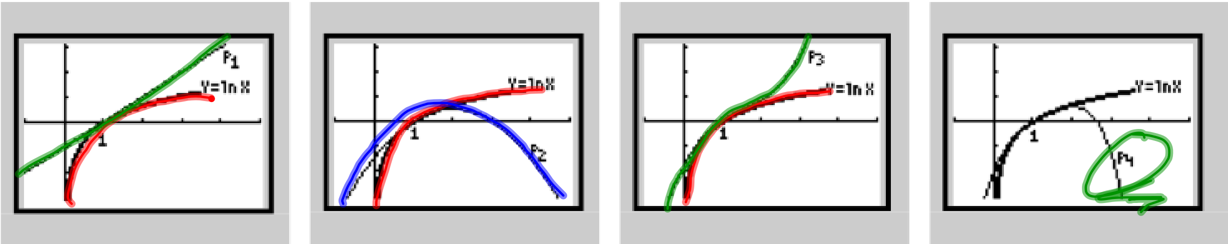
$f''(x) = -x^{-2}$, $f''(1) = -1$

$f'''(x) = 2x^{-3}$, $f'''(1) = 2$

① $P_1 = 0 + 1(x-1)$

② $P_2 = 0 + 1(x-1) + \frac{-1}{2!}(x-1)^2$

③ $P_3 = 1(x-1) - \frac{(x-1)^2}{2!} + \frac{2}{3!}(x-1)^3$



around $\underline{x=c}$

$$f(x) \approx P_n(x) = f(c) + f'(c)(x-c) +$$

$$+ \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} +$$

$$\frac{f^{(4)}(c)(x-c)^4}{4!} + \frac{f^{(5)}(c)(x-c)^5}{5!} + \cdots$$