10.4 Logistic Model

$\frac{dy}{dx} = y' = ky$ — 1st order Diff Eq

$\int \frac{dy}{y} = \int k\,dx$

$\ln y = kx + c$

$y = e^{kx+c}$

$y = e^{c} e^{kx}$

when $x = 0$, $y = y_0$

$y_0 = e^{c} \cancel{e^{k(0)}}$

$y = y_0 e^{kx}$

$y' = ky\left(1 - \frac{y}{L}\right)$

$\frac{dy}{dx} = ky - \cancel{\frac{ky^2}{L}}$

$y << L$ exp

$y \to L$ growth slows down & almost stops

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$y' = ky\left(1 - \frac{y}{L}\right)$

$\frac{dy}{dx} = \frac{ky}{L}(L - y)$

$\int \frac{L\,dy}{y(L-y)} = \int k\,dx = kx + c$

$\frac{A}{y} + \frac{B}{L-y} = \frac{L}{y(L-y)}$

$A(L-y) + B(y) = L$

$y = 0$	$A(L-0) = L$; $A = 1$
$y = L$	$B \cdot L = L$; $B = 1$

$\int \frac{1}{y} + \int \frac{1}{L-y} = kx + c$

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$\int \frac{1}{y}\,dy + \int \frac{1}{L-y}\,dx = kx + c$

$\ln|y| + (-\ln|L-y|) = kx + c$

$\Rightarrow -\ln\left|\frac{y}{L-y}\right| = -(kx + c)$

$\ln\left|\frac{y}{L-y}\right|^{-1} = -kx - c$

$e^{\ln\left|\frac{L-y}{y}\right|} = e^{-kx-c}$

$\frac{L-y}{y} = e^{-kx-c}$

$\frac{L-y}{y} = e^{-c} \cdot e^{-kx}$ — $x = 0$, $y = y_0$

$\frac{L-y}{y} = \frac{L-y_0}{y_0} e^{-kx} \to A$

$\frac{L-y}{y} = A e^{-kx}$

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$\frac{L-y}{y} = Ae^{-kx}$

$\frac{L}{y} - \frac{y}{y} = Ae^{-kx}$

$\frac{L}{y} - 1 = Ae^{-kx}$

$\frac{y}{Ae^{-kx}+1} \cdot \frac{L}{y} = \frac{Ae^{-kx}+1}{1} \cdot \frac{y}{Ae^{-kx}+1}$

$\boxed{\frac{L}{1 + Ae^{-kx}} = y}$

$A = \frac{L - y_0}{y_0}$

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if $y' = ky\left(1 - \frac{y}{L}\right)$

then $y = \frac{L}{1 + Ae^{-kx}}$

where $A = \frac{L - y_0}{y_0}$

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