Taylor's Formula with remainder

- the Lagrange remainder and
-Error Analysis

How well

some $P_n(x)$

approximation of $f(x)$

works!

$p_n(x)$ approximates $f(x)$ closely, but their is some difference.

$R_n(x)$ is the remainder that, when added to $p_n(x)$ lets $p_n(x)=f(x)$.

$$f(x) = p_n(x) + R_n(x)$$

$$\text{Error} = |R_n(x)| = |f(x) - p_n(x)|.$$

$\boxed{R_n(x)}$

The equation for the Lagrange remainder is given below:

$$|R_n(x)| = |f(x) - P_n(x)|$$

$(x-c)^n$

Taylor's Formula with Remainder

Let f be a function whose $(n+1)$st derivative, $f^{(n+1)}(x)$, exists for each x in an open interval I containing c. Then for each x in I,

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(x-c)}{2!}x^2 + \cdots + \frac{f^{(n)}(x-c)}{n!}x^n + R_n(x) \qquad \textbf{Taylor's Formula}$$

where the remainder $R_n(x)$ (or error) is given by the formula

$$\boxed{R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{(n+1)}} \qquad \textbf{The Lagrange Remainder}$$

where z is some number between x and c.

$$\frac{f^{(n+1)}(z)\cdot(x)^{n+1}}{(n+1)!}$$

THE LAGRANGE REMAINDER:

n -> number of derivatives
z -> some number between c and x.

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} x^{(n+1)}$$

Another representation of error: Taylor's Inequality

$$M = \left| f^{(n+1)}(x) \right|_{\max,\ [c,b]}$$

Taylor's Inequality

Suppose $p_n(x)$ is the nth-degree polynomial approximation for the function f about $x = c$ and M is the maximum value of $\left|f^{(n+1)}(x)\right|$ on the interval $[c, b]$ (or $[b, c]$ if $b < c$). Then the error in using the polynomial value $p_n(b)$ to estimate $f(b)$ is bounded by $\frac{M}{(n+1)!}|b-c|^{n+1}$. That is the remainder $R_n(x)$ in Taylor's Formula satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|b-c|^{n+1}.$$

Let f be a function with 5 derivatives on the interval [2, 3] and assume that $|f^{(5)}(x)| < 0.2$ for all x in the interval [2, 3]. If a fourth-degree Taylor polynomial for f at $c = 2$ is used to estimate $f(3)$ how accurate is this approximation?

$M = .2$

Taylor's Inequality

Suppose $p_n(x)$ is the nth-degree polynomial approximation for the function f about $x = c$ and M is the maximum value of $|f^{(n+1)}(x)|$ on the interval $[c, b]$ (or $[b, c]$ if $b < c$). Then the error in using the polynomial value $p_n(b)$ to estimate $f(b)$ is bounded by $\frac{M}{(n+1)!}|b-c|^{n+1}$. That is the remainder $R_n(x)$ in Taylor's Formula satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|b-c|^{n+1}.$$

$$|R_4(x)| \le \frac{.2}{(4+1)!}(1)^5$$

$$R_4(x) \le \frac{.2}{5!} = \frac{.2}{120} = \frac{1}{600}$$

Solution:

Using Taylor's Inequality with $n = 4$, $b = 3$ and $c = 2$ we obtain

$$|R_n(x)| \le \frac{M}{(n+1)!}|b-c|^{n+1} = \frac{M}{5!}|3-2|^5$$

where M is the maximum value of the fifth derivative of f on the closed interval [2, 3]. Because $|f^{(5)}(x)| < 0.2$, we take $M = 0.2$ and obtain

$$|R_n(x)| \le \frac{0.2}{120} \cdot 1^5 = \frac{0.2}{120} < 0.00167 .$$

Thus, the error (the difference between approximation $p_4(3)$ and the actual value $f(3)$) is less than 0.00167 .

Example 2: *Determining the Accuracy of an Approximation*

$n = 3$

Estimate the error that results when $\sin x$ is replaced by $x - \frac{1}{6}x^3$ for $|x| < 0.2$. Support the answer graphically.

$$|R_n(x)| \le \frac{M}{(n+1)!}|b-c|^{n+1}$$

$$|R_3(x)| \le \frac{1}{(3+1)!}|.2|^4$$

$$\le .000067$$

$f^4(x)$

$f = \sin(x)$

$f' \quad \cos x$

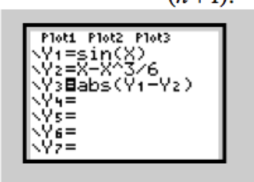
$f'' = -\sin$

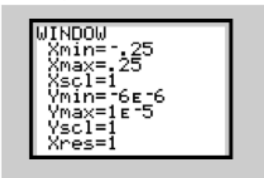
$f^{(4)} = -\cos(x)$

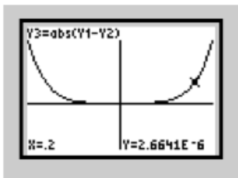
$= 1$

Since $f(x) = \sin x$, the successive derivatives are $\cos x$, $-\sin x$, and $-\cos x$. Because $|\cos x| \le 1$ and $|\sin x| \le 1$, this means we have a simple estimate for the nth derivative of f, namely $|f^{(n)}(x)| \le 1$ for all n and x. Now using Taylor's Inequality with $M = 1$, $b = \pm 0.2$, $n = 3$ and $c = 0$ we have

$$|R_n(x)| \le \frac{M}{(n+1)!}|b-c|^{n+1} \le \frac{1}{4!}|\pm 0.2|^4 = 0.000067.$$







Using a calculator, estimate the values of x for which the Taylor polynomial $\boxed{p_3(x) = x - \frac{x^3}{3}}$ approximates $f(x) = \arctan x$ to four decimal places.

$$\left| f(x) - P_3(x) \right| \le 0.00005$$

y_1 $\qquad$ y_2 $\qquad$ 5E-5

$$X = (-.191, .191)$$