Convergence for Alternating Series

A series in which terms are alternately positive and negative is an **alternating series**. Here are two examples.

$a_{n+1} < a_n$

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \quad \text{(First term is positive.)}$$

$$\sum_{n=1}^{\infty} (-1)^{n}\frac{1}{n!} = -1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots \quad \text{(First term is negative.)}$$

Alternating Series Test

If $a_n > 0$, then an alternating series

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n+1} a_n$$

converges if both of the following conditions are satisfied:

1) $\lim_{n\to\infty} a_n = 0$;
2) $\{a_n\}$ is a decreasing sequence; that is, $a_{n+1} < a_n$ for all n.

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Is the alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

convergent or ~~divergent~~?

Alternating Series Test

If $a_n > 0$, then an alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges if both of the following conditions are satisfied:

1) $\lim_{n\to\infty} a_n = 0$;
2) $\{a_n\}$ is a decreasing sequence; that is, $a_{n+1} < a_n$ for all n.

$\lim_{n\to\infty} (-1)^{n+1}\frac{1}{n}$

$= 0$ ✓

$\frac{1}{2} < 1 \quad \frac{1}{3} < \frac{1}{2} \quad \frac{1}{4} < \frac{1}{3}$

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Error Estimate for Alternating Series

Suppose an alternating series $\sum_{n=1}^{\infty} a_n$ satisfies the conditions of the Alternating Series Test; namely,

$$\lim_{n\to\infty} a_n = 0 \text{ and } \{a_n\} \text{ is a decreasing sequence } (a_{n+1} < a_n).$$

If the series has sum S, then

$$|S - S_n| < a_{n+1}$$

where S_n is the nth partial sum of the series.

$|S - S_n| < a_{n+1}$

S: actual sum

S_n: sum approximation for $n = n$

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Given the convergent alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \quad ✓ \quad a_n = (-1)^{n+1}\frac{1}{n}$$

a) Estimate the sum of the series by taking the sum of the first four terms. How accurate is this estimate?

b) How many terms of this series would we need to take in order to get a partial sum S_n within 0.005 (two decimal places) of the sum S.

Error Estimate for Alternating Series

Suppose an alternating series $\sum_{n=1}^{\infty} a_n$ satisfies the conditions of the Alternating Series Test; namely, $\lim_{n\to\infty} a_n = 0$ and $\{a_n\}$ is a decreasing sequence ($a_{n+1} < a_n$). If the series has sum S, then $|S - S_n| < a_{n+1}$ where S_n is the nth partial sum of the series.

estimate $\boxed{S}$

$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$

$= \frac{12}{12} - \frac{6}{12} + \frac{4}{12} - \frac{3}{12} = \boxed{\frac{7}{12} = S_4}$

$|S - S_4| < a_{4+1}$

$a_5 = (-1)^{(5+1)}\frac{1}{5} = \frac{1}{5}$

$\left|S - \frac{7}{12}\right| < \frac{1}{5}$

$\frac{7}{12} - \frac{1}{5} < S < \frac{7}{12} + \frac{1}{5}$

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Given the convergent alternating harmonic series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

a) Estimate the sum of the series by taking the sum of the first four terms. How accurate is this estimate?

b) How many terms of this series would we need to take in order to get a partial sum S_n within 0.005 (two decimal places) of the sum S.

$|S - S_n| < \underbrace{0.005}_{a_{n+1}}$

$a_{n+1} \le .005$

$\frac{1}{n+1} \le .005$

$\frac{1}{.005} \le n+1$

$200 \le n+1$

$\boxed{n \ge 199}$

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a) Let $a_n = \frac{1}{n}$ and let S denote the actual sum of the series. The error estimate tells us that

$$|S - S_4| \le a_5$$

where S_4 is the sum of the first four terms of the series. Using a calculator, we find

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.583333$$

and $a_5 = \frac{1}{5} = 0.2$. Thus if we estimate S by $S_4 \approx 0.583333$, we incur an error of 0.2, which means that

$$|S - S_4| \le 0.2$$
$$0.583333 - 0.2 \le S \le 0.583333 + 2$$
$$0.383333 \le S \le 0.783333$$

b) We want $|S - S_n| \le 0.005$, and this will hold if $a_{n+1} \le 0.005$. Because $a_{n+1} = \frac{1}{n+1}$, we require $\frac{1}{n+1} \le 0.005$, which is satisfied if $n \ge 199$. Thus, we need 199 terms to get two decimal places of accuracy. This gives you an idea of how slowly the alternating harmonic series converges. The sum of the first 199 terms done on a calculator would be

$$\sum_{n=1}^{199} (-1)^{n+1}\frac{1}{n} = 0.695653$$

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Alternating Series

ABSOLUTE VS. CONDITIONAL CONVERGENCE

Consider the following two series: $(-1)^n$ or $(-1)^{n+1}$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2} = \frac{1}{1}-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}$

Do they converge?

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$ ☑

$a_{n+1} < a_n$ ☑

$\lim_{n\to\infty}\frac{1}{n^2} = 0$ ☑

$a_{n+1} < a_n$ ☑

Converge

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$\sum(-1)^{n+1}\frac{1}{n}$ $\sum(-1)^{n+1}\frac{1}{n^2}$

What if we took the Absolute Value?

converges conditionally

$\sum\left|(-1)^{n+1}\frac{1}{n}\right|$

$= \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots$

$\int\frac{1}{x}\,dx$ $p=1$

$p\leq 1$

diverge

Converges Conditionally

$\int_1^{\infty}\frac{1}{n^2} = -n^{-1}\Big|_1^{\infty}$

$= -\frac{1}{\infty}-\left(-\frac{1}{1}\right)$

$= 1$

$\sum\left|(-1)^{n+1}\frac{1}{n^2}\right|$

$\sum\frac{1}{n^2}$ p-series with $p=2$

$2>1$

Converge

Converges absolutely

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Absolutely Convergent Series

The series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum|a_n|$ converges.

Conditionally Convergent Series

The series $\sum a_n$ is called **conditionally convergent** if it converges but the series of absolute values $\sum|a_n|$ diverges

Absolute Convergence Implies Convergence

If the series $\sum|a_n|$ converges, then $\sum a_n$ converges.

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Determine whether each of the following series of constants is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a) $\sum_{n=1}^{\infty}(-1)^n\frac{2n^2}{n^2+10n} = -\frac{2}{11}+\frac{1}{3}-\frac{6}{13}+\frac{4}{7}-\frac{2}{3}+\cdots$ Divergent

b) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}} = -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\cdots$

c) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = 1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots$

a) $\lim_{n\to\infty} a_n$

$\lim_{n\to\infty}\frac{2n^2}{n^2+10} = 2 \neq 0$

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b) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}} = -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\cdots$

c) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = 1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots$

$a_{n+1} < a_n$ ☑

$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}} = 0$ ☑

conditionally convergent

$\int_1^{\infty}\frac{1}{\sqrt{x+1}}\,dx = \int_1^{\infty}(x+1)^{-1/2}\,dx$

$\lim_{a\to\infty} 2(x+1)^{1/2}\Big|_1^a$

$= \lim_{a\to\infty}\left[2\sqrt{(a+1)}-2\sqrt{1+1}\right]$

Diverges

Conditional Convergence

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Determine whether each of the following series of constants is convergent or divergent. Classify any convergent series as absolutely or conditionally convergent.

a) $\sum_{n=1}^{\infty}(-1)^n\frac{2n^2}{n^2+10n} = -\frac{2}{11}+\frac{1}{3}-\frac{6}{13}+\frac{4}{7}-\frac{2}{3}+\cdots$

b) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}} = -\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}+\frac{1}{\sqrt{5}}+\cdots$

c) $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^4} = 1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots$

$\lim_{n\to\infty} a_n$

$\lim_{n\to\infty}\frac{1}{n^4} = 0$ ☑

$a_{n+1} < a_n$ ☑

conditionally convergent

$\sum\frac{1}{n^4}$ $p=4$

$p=4>1$

Convergence

Absolutely convergent

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