TESTING FOR CONVERGENCE AT ENDPOINTS

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$|x| < 1$

$\boxed{-1 < x < 1}$

$|x| > 1$

does not converge!

$|x| = 1$??

If a function is monotonic (always increasing or always decreasing) and it is bounded, we have a great criterion.

Convergence Criterion for a Bounded Monotonic Sequence

If a sequence $\{a_n\}$ is bounded and monotonic then it converges to some finite limit L.

$$\lim_{n \to \infty} a_n = L$$

Example 1: *Bounded and Monotonic Sequences*

$\lim_{n\to\infty} \frac{n}{n+1} = 1$, as $n\uparrow$, $a_n \uparrow$

a) The sequence $a_n = \frac{n}{n+1}$ is both bounded and increasing and thus must converge.

b) The divergent sequence $b_n = \frac{n^2}{n+1}$ is increasing but not bounded above.

$\lim_{n\to\infty} \frac{n^2}{n+1} = \infty$

c) The divergent sequence $c_n = (-1)^n$ is bounded but not increasing.

$-1, 1, -1, 1, -1, 1$

In view of this convergence criterion, deciding whether a series with nonnegative terms is convergent or divergent means we need to determine whether the increasing sequence of partial sums $\{S_n\}$ is bounded from above.

The Integral Test

$f(n)$

Let f be a decreasing continuous function that has positive values $f(x)$ for all $x \geq 1$.

The infinite series $\sum_{n=1}^{\infty} f(n)$

a) converges if the improper integral $\int_1^{\infty} f(x)\,dx$ converges;

b) diverges if the improper integral $\int_1^{\infty} f(x)\,dx$ diverges.

$$\boxed{\sum_{n=1}^{\infty} f(n)} < \boxed{\int_1^{\infty} f(n)\,dn}$$

$$L = \int_1^{\infty} f(x)\,dx$$

If the improper integral $\int_1^{\infty} f(x)\,dx$ converges to a limit L, then from the right-hand inequality above

$$S_n < a_1 + \int_1^{n} f(x)\,dx < a_1 + \int_1^{\infty} f(x)\,dx = a_1 + L$$

$$S_n < a_1 + L$$

Use the Integral Test to show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$$S_1 = \frac{1}{1}$$

$$S_2 = \frac{1}{1} + \frac{1}{2}$$

$$S_3 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3}$$

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{a \to \infty} \int_1^{a} \frac{1}{x}\,dx$$

$$\lim_{a \to \infty} \ln|x| \Big|_1^{a}$$

$$= \lim_{a \to \infty} \left[\ln|a| - \cancel{\ln|1|} \right]$$

$$= \text{DNE}$$

The harmonic series $\sum \frac{1}{n}$ is a special case of a class of series called **p-series.** A p-series is an infinite series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

where $p > 0$. If we define $f(x) = \frac{1}{x^p}$, then f meets the criteria of the Integral Test, and

$$\int_1^t \frac{1}{x^p} = \begin{cases} \ln t & \text{if } p = 1 \\ \dfrac{t^{(-p+1)} - 1}{-p+1} & \text{if } p \neq 1 \end{cases}$$

$p = 1$

$$\int_1^t \frac{1}{x^1} = \ln|x| \Big|_1^t = \ln t - \ln 1 = \ln t$$

for $p \neq 1$

$$\lim_{t \to \infty} \int_1^t \frac{1}{x^p}\,dx = \frac{x^{-p+1}}{-p+1} \Big|_1^t = \frac{t^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1} = \frac{t^{-p+1} - 1}{-p+1}$$

p-series Test

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Determine whether the following series converge or diverge.

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\frac{1}{n^{1/2}}$

$p = 1/2$

$1/2 < 1$

diverge

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

$\frac{1}{n^{3/2}}$

$p = 3/2$

$3/2 > 1$

converge

$\lim_{t\to\infty} \int_1^t \frac{1}{x^{3/2}}\,dx$

$= \lim_{t\to\infty} \int_1^t x^{-3/2}\,dx$

$\left. \frac{x^{-1/2}}{-1/2} \right|_1^t$

$\lim_{t\to\infty} \left[-2t^{-1/2} - \left(-2(1)^{-1/2}\right) \right]$

$0 + 2 = \boxed{2}$

$\lim_{t\to\infty} \int_1^t \frac{1}{x^{1/2}}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/2}\,dx$

$\left. 2x^{1/2} \right|_1^t = \lim_{t\to\infty} \left[2t^{1/2} - 2 \right]$