# POWER SERIES APPROXIMATIONS

Find a power series, centered at $x = 0$, for $f(x) = xe^{-2x}$

$$e^{(x)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{(-2x)} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!}$$

$$= 1 - 2x + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!}$$

$$x \cdot e^{(-2x)} = x \cdot \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n \cdot x^n}{n!}\right)$$

$n=3$

$$\frac{(-1)^3 \cdot 2^3 x^3}{3!}$$

$$xe^{-2x} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n x^n \cdot x}{n!}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n x^{n+1}}{n!}\right)$$

Converges for all $x$.

First we substitute $-2x$ in the power series for $e^x$.

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} - \frac{(-2x)^3}{3!} + \cdots + \frac{(-2x)^n}{n!} + \cdots,$$

or $e^{-2x} = 1 - 2x + 2^2 \dfrac{x^2}{2!} - 2^3 \dfrac{x^3}{3!} + \cdots + (-2)^n \dfrac{x^n}{n!} + \cdots$

which can be written as

$$e^{-2x} = \sum_{n=0}^{\infty} (-2)^n \frac{x^n}{n!}.$$

Multiplying both sides by $x$ gives us

$$xe^{-2x} = x - 2x^2 + 2^2 \frac{x^3}{2!} + 2^3 \frac{x^4}{3!} + \cdots + (-2)^n \frac{x^{n+1}}{n!} + \cdots$$

$$xe^{-2x} = \sum_{n=0}^{\infty} (-2)^n \frac{x^{n+1}}{n!} \quad \text{for all } x.$$

| Function | Series | Radius of Convergence |
|---|---|---|
| $e^x$ | $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots$ | $R=\infty$ |
| $\sin x$ | $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$ | $R=\infty$ |
| $\cos x$ | $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots$ | $R=\infty$ |
| $\frac{1}{1-x}$ | $1+x+x^2+x^3+x^4+\cdots$ | $R=1$ |
| $\cancel{\arctan x}$ | $\cancel{x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots}$ | $\cancel{R=1}$ |