Representing Functions with Power Series

Allows us to see whether functions converge and what the radius of convergence is.

Function	Series	Radius of Convergence
e^x	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$	$R = \infty$
$\sin x$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$	$R = \infty$
$\cos x$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$	$R = \infty$
$\frac{1}{1-x}$	$1 + x + x^2 + x^3 + x^4 + \cdots$	$R = 1$
$\cancel{\arctan x}$	$\cancel{x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots}$	$\cancel{R = 1}$

Geometric Power Series

The series

$$a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

converges to $\frac{a}{1-r}$ if $|r| < 1$. Using this formula and a little algebraic ingenuity we can produce many other useful series.

Example: Find the convergence for f(x)=1/(1+x)

Maclaurin Series (x=0)

$$\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 +$$

$$\sum_{n=0}^{\infty} (-1)^n \cdot x^n$$

$n=1,\ -1x^1$

$n=2\ (-1)^2 x^2$

$|-x| < 1$; $R = 1$

$-1 < x < 1$

Find a power series centered at $x = 1$ for the function $f(x) = \dfrac{5}{3-x}$. What is its radius of convergence?

$\dfrac{a_0}{1-r}$

$$\frac{5}{3-[(x-1)+1]} = \frac{5}{3-(x-1)-1}$$

$$= \frac{\frac{1}{2}\,5}{\frac{1}{2}\left(2-(x-1)\right)} = \frac{5/2}{1-\frac{(x-1)}{2}}$$

$$\boxed{\frac{5}{2} + \frac{5}{2}\left(\frac{x-1}{2}\right) + \frac{5}{2}\left(\frac{x-1}{2}\right)^2 + \frac{5}{2}\left(\frac{x-1}{2}\right)^3}$$

$$\left|\frac{x-1}{2}\right| < 1$$

$$|x-1| < 2 \qquad R = 2$$

$$\boxed{-1 < x < 3}$$

$$\sum_{n=0}^{\infty} \frac{5}{2}\left(\frac{x-1}{2}\right)^n$$

Find a Taylor series representation for the function $f(x) = e^{-x^2}$ by substitution.

$$e^{x} \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{(-x^2)} \approx 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!}$$

$$= 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!}$$

$$\sum_{n=0}^{\infty} (-1)^n \left(\frac{x^{2n}}{n!}\right)$$

Find a Taylor series that represents $f(x) = \cos\sqrt{x}$.

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +$$

$$\cos(\sqrt{x}) = 1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \dots$$

$$= 1 - \frac{x^1}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \dots$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{(2n)!}$$; converge for $x \geq 0$

Starting with the series $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$

and substituting $\sqrt{x}$ for x we obtain the (alternating) series

$$\cos\sqrt{x} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \frac{x^4}{8!} + \cdots$$

This series converges for all x in the domain of $\cos\sqrt{x}$; that is, for $x \geq 0$.

If the power series $\sum a_n(x-c)^n$ has radius of convergence $R>0$, then the function *f* defined by

$$f(x)=\sum_{n=0}^{\infty} a_n(x-c)^n,$$

is differentiable on the interval $(c-R, c+R)$ with

(a) $f'(x)=a_1+2a_2(x-c)+3a_3(x-c)^2+\cdots=\sum_{n=1}^{\infty} na_n(x-c)^{n-1}$ and

(b) $\int f(x)\,dx = C+a_0(x-c)+a_1\dfrac{(x-c)^2}{2}+a_2\dfrac{(x-c)^3}{3}+\cdots \;=\; C+\sum_{n=0}^{\infty}\dfrac{a_n(x-c)^{n+1}}{n+1}.$

The series obtained by differentiating in (a) or integrating in (b) has the same radius of convergence as the original series $\sum a_n(x-c)^n$.

Find a power series representation for $\dfrac{1}{(1-x)^2}$ by differentiation. What is its radius of convergence?

$$\frac{d}{dx}\frac{1}{1-x} = -(1-x)^{-2}(-1) \qquad R=1$$

$$= \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}\left\{\frac{1}{1-x}\right\} = \frac{d}{dx}\left(1+x+x^2+x^3+x^4+\dots\right)$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+5x^4$$

$$\sum_{n=1}^{\infty} n\cdot x^{n-1} \qquad R=1$$

By integrating an appropriate geometric series find a power series representation for $f(x) = \ln(1+x)$ and its radius of convergence.

$R = 1$

$\int \frac{1}{1+x} = \ln|1+x| + c$

$\int \frac{1}{1+x} = \int 1 - x + x^2 - x^3 + x^4 - x^5$

$\ln|1+x| = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 + C$

$\underbrace{\ln(1)}_{=0} = 0 - 0 + 0 - 0 + 0 - 0 \ldots + \cancel{C}$

$\ln|1+x| = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

$R = 1$

Combining Power Series

Find a power series, centered at $x=0$, for $f(x)=\dfrac{3-4x}{2x^2-3x+1}$

$\dfrac{a_0}{1-r}$

$$\frac{3-4x}{2x^2-3x+1}=\frac{1}{1-x}+\frac{2}{1-2x}$$

$$\frac{A}{(2x-1)}+\frac{B}{(x-1)}$$

$$\frac{1}{1-x}=1+x+x^2+x^3+\dots=\boxed{\sum_{n=0}^{\infty}x^n}$$

for $-1<x<1$

$$\frac{2}{1-2x}=2+4x+8x^2+16x^3+32x^4$$

$$\sum_{n=0}^{\infty}2(2x)^n$$

$|2x|<1$

$|x|<\frac{1}{2}$

$-1/2<x<1/2$

$$\frac{3-4x}{2x^2-3x+1}=\sum_{n=0}^{\infty}x^n+\sum_{n=0}^{\infty}2(2x)^n$$

$(-\frac{1}{2},1/2)$

$$=\sum_{n=0}^{\infty}\left[x^n+2(2x)^n\right]$$

$R=1/2$

$(-1/2,1/2)$